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Interesting Instructional Problems in Column Buckling for the Strength of Materials and Mechanics of Solids Courses*

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In this study the following problem is addressed: a uniform column is subjected to a compressive load; an additional support is placed to increase the buckling load. The following question is posed: where to place the support location so as to maximize and evaluate the resulting buckling load? It turns out that this question can be effectively dealt with in the standard courses of Strength of Materials, Mechanics of Solids, or Mechanics of Materials, since all the necessary tools needed are presently uniformly taught in these existing courses. Including this interesting case into the curriculum may enhance students' grasp of the subject, sharpen their mind, and trigger an additional interest in the exciting subject of theoretical and applied mechanics. Topics covered in sections 1–3 can be taught in one or two 50-minute lectures, whereas the exposition of the entire material may take between two to three 50-minute lectures, depending on the interest of students.

SUMMARY OF EDUCATIONAL ASPECTS OF THE PAPER

1. The paper discusses materials for a course in Strength of Materials and Mechanics of Solids.
2. Students of second-year Mechanical Engineering, Ocean Engineering, Civil Engineering and Aerospace Engineering are taught this course.
3. The mode of presentation is by lecture and is run as a regular course.
4. Hours required to cover the material is 4 to 5 with 2 to 3 revision hours.
5. The novel aspects presented in this paper hopefully help to sharpen the minds of the students and increase interest in applied mechanics.
6. The standard text recommended for the course is Hibbeler, *Mechanics of Materials* [2].

1. INTRODUCTION

NATURALLY THERE IS a big gap between the exposition of the topic of buckling in the undergraduate textbooks devoted to Mechanics of Solids, Strength of Materials or Mechanics of Materials [1–5] and more specialized monographs, suitable for the elective or graduate courses [6–10]. The undergraduate textbooks invariably report buckling of uniform columns with various boundary conditions, whereas the advanced texts include numerous results of uniform or non-uniform columns, plates and shells, pertinent to engineering practice. Among other topics, specialized books report results on optimization of columns, plates and shells under buckling conditions [11, 12]. It appeared to the authors, that the motivation of the students can be enhanced, if some model problems can be included on more advanced topics in the undergraduate courses. This would provide students with some 'raisins to look for'. As Budiansky and Hutchinson [13] note, 'Everyone loves a buckling problem'. We propose some problems the solutions to which hopefully will enlarge the number of 'buckling lovers,' to include more undergraduate students. Maybe even every undergraduate textbook should include a section on 'Some Interesting Problems' providing, gastronomically speaking, the dessert after the main serving of essential material is taught. We deal here with the problem of locating the intermediate support in a uniform column with ideal boundary conditions so as to maximize the buckling load. The analysis is conducted via the straightforward optimization. It turns out that with the knowledge acquired by the undergraduate students in the standard course, some interesting problems can be solved, even those with seemingly un-intuitive solutions.

Note that the beams and columns with intermediate supports have been dealt with by Olhoff and Taylor [14], Rozvany and Mroz [15], Wang, *et al.* [16–19], Liu, *et al.* [20, 21], and Hou [22] in various optimization contexts. The above studies used Bernoulli-Euler theory. Ari-Gur and Elishakoff [23] studied the effect of shear deformation on buckling of columns with intermediate support. Here we deal with the instructional side of this problem for uniform columns with different boundary conditions.

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2. UNIFORM COLUMN SIMPLY-SUPPORTED AT BOTH ENDS, WITH AN INTERMEDIATE SUPPORT

The buckling of a uniform column is governed by the following differential equation:

$$EI\,\frac{d^4 w}{dx^4} + P\,\frac{d^2 w}{dx^2} = 0 \tag{1}$$

where $w(x)$ is the transverse displacement, $x =$ axial coordinate, $E =$ modulus of elasticity, $I =$ moment of inertia, $P =$ compressive loading. We divide equation (1) by flexural stiffness EI, and denote:

$$\frac{P}{EI} = k^2 \tag{2}$$

to obtain:

$$\frac{d^4 w}{dx^4} + k^2\,\frac{d^2 w}{dx^2} = 0 \tag{3}$$

At the cross-section $x = a$, an additional support is placed (Fig. 1a). Thus, two regions are created. For the purpose of identification the displacement in the first region is denoted by $w_1(x)$, whereas the displacement in the second region ($a < x \leq L$), is denoted by $w_2(x)$. Thus, instead of equation (3) we have the following two equations:

$$\frac{d^4 w_1}{dx^4} + k^2\,\frac{d^2 w_1}{dx^2} = 0, \quad \text{for } 0 \leq x < a \tag{4}$$

$$\frac{d^4 w_2}{dx^4} + k^2\,\frac{d^2 w_2}{dx^2} = 0, \quad \text{for } a \leq x \leq L \tag{5}$$

The boundary conditions read:

$$w_1(0) = w_1''(0) = 0 \tag{6}$$

$$w_2(L) = w_2''(L) = 0 \tag{7}$$

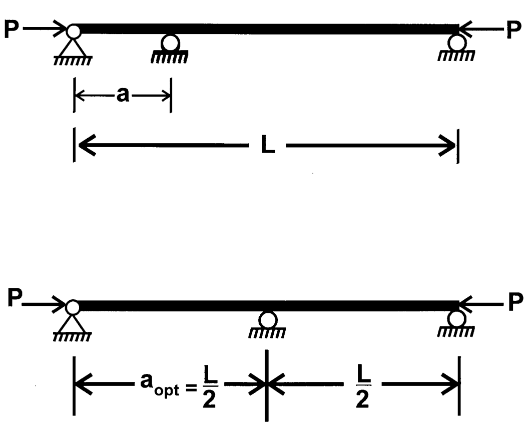

Fig. 1. (a) Uniform column, simply supported at its ends with an intermediate support. (b) Optimal location of an intermediate support is in column's middle.

The continuity conditions read:

$$w_1(a) = w_2(a) = 0 \tag{8}$$

$$w_1'(a) = w_2'(a) \tag{9}$$

$$w_1''(a) = w_2''(a) \tag{10}$$

Equation (9) signifies the continuity of the slopes, whereas equation (10) indicates the continuity of bending moments.

The general solution $w_1(x)$ in the first region is:

$$w_1(x) = A_1 \sin kx + A_2 \cos kx + A_3 x + A_4 \tag{11}$$

Satisfying the conditions of equation (6) and the first of the conditions in equation (8) we get:

$$w_1(x) = A_1\left(\sin kx - \frac{x}{a}\sin ka\right) \tag{12}$$

The general solution $w_2(x)$ in the second region is:

$$w_2(x) = B_1 \sin kx + B_2 \cos kx + B_3 x + B_4 \tag{13}$$

where β_j's are the constants of integration. After satisfaction of the conditions in equation (7) and the second condition, equation (8), equation (13) reduces to:

$$w_2(x) = B_1\left[\sin k(L-x) - \frac{L-x}{L-a}\sin k(L-a)\right] \tag{14}$$

Equation (9) results in:

$$A_1\left(k\cos ka - \frac{\sin ka}{a}\right) + B_1\left[k\cos k(L-a) - \frac{\sin k(L-a)}{L-a}\right] = 0 \tag{15}$$

Equation (10) yields:

$$A_1 \sin ka - B_1 \sin k(L-a) = 0 \tag{16}$$

Non-triviality of A_1 and B_1 yields the transcendental equation:

$$\sin ka\left[k\cos k(L-a) - \frac{\sin k(L-a)}{L-a}\right] + \sin k(L-a)\left(k\cos ka - \frac{1}{a}\sin ka\right) = 0 \tag{17}$$

Introducing non-dimensional quantities:

$$u = \frac{a}{L}, \quad \alpha = kL, \tag{18}$$

and using trigonometric identities we arrive at the following characteristic equation:

$$2\alpha \sin\alpha + \frac{1}{u-u^2}\left[\cos\alpha - \cos\alpha(2u-1)\right] = 0 \tag{19}$$

Now we turn to determining a location $a = a^*$ of the intermediate support, such that the buckling load will attain a maximum at $a = a^*$ corresponds to $u = a^*/L = u^*$. We first note that one of the arguments on the left hand side of equation (19), denoted by f:

$$f(\alpha(u), u) = 2\alpha \sin \alpha + \frac{1}{u - u^2}[\cos \alpha - \cos \alpha(2u - 1)] \quad (20)$$

namely $\alpha(u)$, is an implicit function of u. We wish to determine the derivative $\mathrm{d}\alpha/\mathrm{d}u$ and set it equal to zero. We proceed as follows. Differentiating equation (19) with respect to u, we get:

$$\frac{\partial f}{\partial \alpha}\frac{\mathrm{d}\alpha}{\mathrm{d}u} + \frac{\partial f}{\partial u} = 0 \quad (21)$$

Thus,

$$\frac{\mathrm{d}\alpha}{\mathrm{d}u} = -\left(\frac{\partial f}{\partial u}\right) \Big/ \left(\frac{\partial f}{\partial \alpha}\right) \quad (22)$$

On the assumption that $\partial f/\partial \alpha$ differs from zero, we see that the condition:

$$\frac{\mathrm{d}\alpha}{\mathrm{d}u} = 0 \quad (23)$$

is equivalent to:

$$\frac{\partial f}{\partial u} = 0 \quad (24)$$

The latter equation yields:

$$-\frac{1 - 2u}{(u - u^2)^2}[\cos \alpha - \cos \alpha(2u - 1)] + \frac{2\alpha}{(u - u^2)} \sin \alpha(2u - 1) = 0 \quad (25)$$

We are interested in the smallest non-zero solution for α of the coupled set of equations (19) and (25), and call it α_{cr}; the corresponding value of u is called u_{cr}.

Let us examine equation (25). It is satisfied by $u^* = a^*/L = 1/2$ independent of the value of α. Hence, by putting $u = u^* = 1/2$ into equation (19) we get:

$$\sin \frac{\alpha}{2}\left(\alpha \cos \frac{\alpha}{2} - 2 \sin \frac{\alpha}{2}\right) = 0 \quad (26)$$

Equation (26) yields two equations. Either:

$$\sin \frac{\alpha}{2} = 0 \text{ with } \alpha = 2\pi \quad (27)$$

or,

$$\tan \frac{\alpha}{2} = \frac{\alpha}{2} \text{ with } \alpha \simeq 8.986 \quad (28)$$

The smaller value is $\alpha = 2\pi$, resulting in the buckling load:

$$P_{cr} = \alpha_{cr}^2 \frac{EI}{L^2} = \frac{4\pi^2 EI}{L^2} \quad (29)$$

We conclude that the best location for the intermediate support, if the column is simply supported at both ends is the middle cross section (Fig. 1b), with the buckling load given in equation (29). Let us consider a limiting case.

3. BUCKLING LOAD OF A SIMPLY-SUPPORTED COLUMN WITH INTERMEDIATE SUPPORT'S LOCATION APPROACHING ZERO

Taking into account the expressions for the transverse displacement $w_1(x)$ and $w_2(x)$ given in equations (12) and (14), respectively, we obtain the following formulas for the slopes at the intermediate support location:

$$w_1'(a) = A_1\left(k \cos ka - \frac{\sin ka}{a}\right) \quad (30)$$

$$w_2'(a) = B_1\left[k \cos k(L - a) - \frac{1}{L - a} \sin k(L - a)\right] \quad (31)$$

Now, when a approaches zero (Fig. 2a), we get:

$$\begin{aligned}\lim_{a \to 0} w_1'(a) &= A_1 \lim_{a \to 0}\left(k \cos ka - \frac{1}{a} \sin ka\right) \\ &= A_1\left(k \lim_{a \to 0} \cos ka - \lim_{a \to 0} \frac{1}{a} \sin ka\right) \\ &= A_1(k - k) = 0 \end{aligned} \quad (32)$$

Since condition (9) requires $w_1'(a) = w_2'(a)$ for all a, we thus also require that:

$$\lim_{a \to 0} w_2'(a) = 0 \quad (33)$$

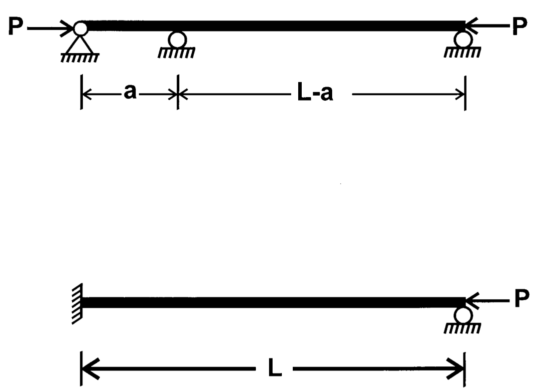

Fig. 2. (a) A simply-supported column with intermediate support's location approaching zero. (b) Within the Bernoulli-Euler theory, when an intermediate support approaches the left end, the boundary condition there tends to that of the clamped end.

Thus,

$$\lim_{a\to 0} w_2'(a)$$

$$= B_1 \lim_{a\to 0}\left[k\cos k(L-a) - \frac{1}{L-a}\sin k(L-a)\right]$$

$$= B_1\left(k\cos kL - \frac{1}{L}\sin kL\right) = 0 \quad (34)$$

or,

$$\tan kL = kL \quad (35)$$

which is the characteristic equation of the column that is clamped at the left end and simply supported at the right end.

Thus, within the Bernoulli-Euler theory, when the intermediate support approaches the left end, the boundary condition at the left support tends to that of the clamped end (Fig. 2b). Note that to the students who will be surprised with this conclusion, the lecturer should explain that within the more refined Timoshenko beam theory this 'phenomenon' does not take place: when two supports tend to each other, they again make the simply support condition, as expected. We observe that a more difficult theory leads to the result that can be 'seen' immediately, whereas the simplified theory of Bernoulli-Euler leads to a less 'digestible' result. This problem is addressed by Ari-Gur and Elishakoff [23].

4. UNIFORM COLUMN CLAMPED AT BOTH ENDS, WITH AN INTERMEDIATE SUPPORT

Consider now a uniform column that is clamped at both ends and is supported by an additional support at the location $x = a$ (Fig. 3a). We again have two regions, with attendant governing differential equations as per equations (4) and (5).

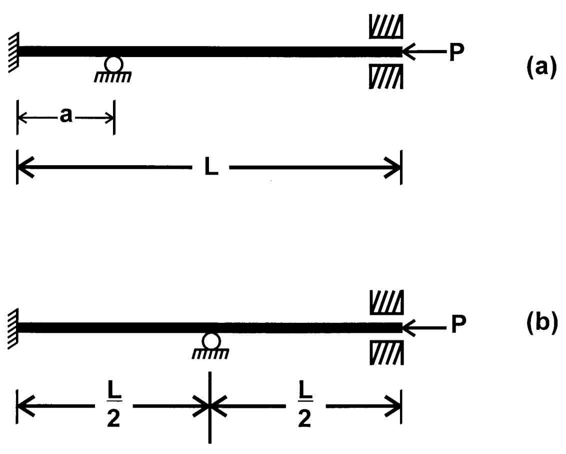

Fig. 3. (a) Uniform column clamped at its ends with an intermediate support. (b) Optimal location of an intermediate support is in column's middle.

The boundary conditions read, in the new circumstances:

$$w_1(0) = w_1'(0) = 0 \quad (36)$$

$$w_2(L) = w_2'(L) = 0 \quad (37)$$

whereas the continuity conditions remain unchanged, and are given by equations (8–10). In the first region the solution that satisfies the boundary conditions and the first of the equation (8) reads (see Appendix A):

$$w_1(x) = A[-\sin k(x-a) - \sin ka - ka\cos kx + kx\cos ka + \sin kx - k(x-a)] \quad (38)$$

In the second region the transverse displacement becomes (see Appendix B):

$$w_2(x) = C\{k(L-a)\cos k(x-L) + \sin k(x-L) - \sin k(x-a) + \sin k(L-a) - kL\cos k(L-a) + ak + [k\cos k(L-a) - k]x\} \quad (39)$$

where A and C are arbitrary constants. The two remaining continuity conditions in equations (9) and (10) yield:

$$A(2k\cos ka + k^2 a\sin ka - 2k) + C[k^2(L-a)\sin k(a-L) - 2k\cos k(a-L) + 2k] = 0 \quad (40)$$

$$(k^3 a\cos ka - k^2\sin ka)A + [k^3(L-a)\cos k(a-L) + k^2\sin k(a-L)]C = 0 \quad (41)$$

Setting the determinant of the system (40) and (41) equal to zero results in the characteristic equation:

$$(2\cos ka + ka\sin ka - 2) \times [k(L-a)\cos k(a-L) + \sin k(a-L)] + (\sin ka - ka\cos ka) \times [k(L-a)\sin k(a-L) - 2\cos k(a-L) + 2] = 0 \quad (42)$$

Using again the notations as per equation (18) transforms the characteristic equation into:

$$(2\cos\alpha u + \alpha u\sin\alpha u - 2) \times [\alpha(1-u)\cos\alpha(1-u) - \sin\alpha(1-u)] + (\sin\alpha u - \alpha u\cos\alpha u) \times [-\alpha(1-u)\sin\alpha(1-u) - 2\cos\alpha(1-u) + 2] = 0 \quad (43)$$

Using trigonometric identities simplifies this equation into:

$$
\begin{aligned}
f(\alpha(u), u) = {} & [\alpha^2 u(1-u) - 2]\sin\alpha \\
& + \alpha\cos\alpha + \alpha\cos\alpha u\cos\alpha(1-u) \\
& - 2\alpha(1-u)\cos\alpha(1-u) \\
& + 2\sin\alpha(1-u) + 2\sin\alpha u \\
& - 2\alpha u\cos\alpha u = 0 \qquad (44)
\end{aligned}
$$

Setting the partial derivative of the left-hand side of equation (44) with respect to u equal to zero yields:

$$
\begin{aligned}
& (1-2u)\alpha^2\sin\alpha + \alpha^2\sin\alpha(1-2u) \\
& \quad - 2\alpha^2(1-u)\sin\alpha(1-u) \\
& \quad + 2\alpha^2 u\sin\alpha u = 0 \qquad (45)
\end{aligned}
$$

It is immediately recognized that this equation is satisfied by $u^* = 1/2$. Substituting $u^* = 1/2$ into the equation (44) we obtain:

$$\alpha = 4\pi \qquad (46)$$

That is, exactly as in the column that is simply supported at both ends, the location of the intermediate support that maximizes the buckling load is in the middle of the column (Fig. 3b). The appropriate buckling load equals:

$$P_{cr} = \frac{16\pi^2 EI}{L^2} \qquad (47)$$

Can we draw a general conclusion from considering these two problems together? It appears that this question should be addressed in the classroom. Some students may suggest that due to the uniformity of the column and the symmetry of the boundary conditions with respect to the middle cross-section, either the maximum or the minimum buckling load should occur when the intermediate support is placed in the middle cross-section. At this point the lecturer may point out that in the case of the column without intermediate support that is simply supported at its ends, the buckling load equals $P_{cr} = \pi^2 EI/L^2$, whereas placement of the support in the middle cross-section changes the buckling load to $P_{cr} = 4\pi^2 EI/L^2$, i.e. increases it four-fold. This suggests that students' anticipation that the placement of the additional support in the middle cross-section makes for the extremes of buckling load is correct. The calculation above, however, shows us that this extremum is a maximum.

The lecturer may ask the students to provide analogous reasoning for the column that is clamped at the ends. At this juncture the lecturer may ask if the following theorem would be correct: 'In order to maximize the buckling load of a uniform column, an intermediate support has to be placed in the middle cross section.' We hope that some students will refute this statement claiming that (1) we have not checked all possible boundary conditions, (2) that presumably for this theorem to be correct one should mention the symmetry in boundary conditions and change the statement into: 'In order to maximize the buckling load of a uniform column with symmetric boundary conditions (i.e. identical boundary conditions) at its ends, an intermediate support should be placed in the middle cross section.'

If the class is sufficiently strong, the lecturer may proceed to discuss additional cases, which will reveal that there are more fundamental considerations pertinent to the buckling of a column with intermediate support, than those associated with the symmetry. The lack of symmetry in the boundary conditions of the column simply supported at one end and clamped at the other should appear sufficient to decide that the intermediate support should be placed elsewhere (not in the middle cross-section) in order to maximize the buckling load.

We also observe that the location of the additional support that makes the buckling load maximum corresponds to the location of the node in the second buckling node of the columns considered in the above two cases. The interest of the second author in this problem arose when he attended the International Conference on Vibration Engineering in Beijing, People's Republic of China, in 1994, and sat next to Prof. H. C. Hu. The presentation of Ref. 20 was made by Dr. Z. S. Liu. During the presentation, which dealt with the vibration problem of the beam, Dr. Liu formulated the following theorem:

> The location of the additional support to maximize the natural frequency of the beam coincides with that of the node of the second vibration mode of the beam without additional support.

The second author asked Professor Hu if one could foresee this interesting theorem. The reply was affirmative: 'Yes, I have dreamt it!'

One may therefore make a conjecture that for the column an analogous theorem would hold:

> The location of the additional support to maximize the buckling load of the column coincides with that of the node of the second buckling mode of the column without support.

Let us check if this conjecture is satisfied for the column that is simply supported at one end and clamped at the other, with an intermediate support.

5. UNIFORM COLUMN SIMPLY SUPPORTED AT ONE END AND CLAMPED AT THE OTHER END, WITH INTERMEDIATE SUPPORT

The boundary conditions read (Fig. 4a):

$$w_1(0) = w_1'' = w_1(a) = 0 \qquad (48)$$

$$w_2(L) = w_2'(L) = w_2(a) = 0 \qquad (49)$$

In the first region $0 \leq x \leq a$, we have:

$$w_1(x) = A\left(\sin kx - \frac{x}{a}\sin ka\right) \tag{50}$$

In region 2, $a \leq x \leq L$, we adopt the solution used for the column that is clamped at both ends:

$$w_2(x) = C[\omega_1 \sin kx + \omega_2 \cos kx + \omega_3 x + \omega_4] \tag{51}$$

where

$$\begin{aligned} \omega_1 &= (L-a)k\sin kL + \cos kL - \cos ka \\ \omega_2 &= (L-a)k\cos kL + \sin ka - \sin kL \\ \omega_3 &= k[\cos k(L-a) - 1] \\ \omega_4 &= \sin k(L-a) - kL\cos k(L-a) + ak \end{aligned} \tag{52}$$

The continuity of bending moment:

$$w_1''(a) = w_2''(a) \tag{53}$$

leads to:

$$-A\sin ka + C(\omega_1 \sin ka + \omega_2 \cos ka) = 0 \tag{54}$$

The imposition of the continuity of the slope:

$$w_1'(a) = w_2'(a) \tag{55}$$

results in:

$$A\left(k\cos ka - \frac{\sin ka}{a}\right) - C(k\omega_1 \cos ka - k\omega_2 \sin ka + \omega_3) = 0 \tag{56}$$

Requirement of non-triviality of $A^2 + C^2$ in equations (54) and (56) results in the characteristic equation:

$$\begin{aligned} &\sin ka(k\omega_1 \cos ka - k\omega_2 \sin ka + \omega_3) \\ &\quad - \left(k\cos ka - \frac{\sin ka}{a}\right) \\ &\quad \times (\omega_1 \sin ka + \omega_2 \cos ka) = 0 \end{aligned} \tag{57}$$

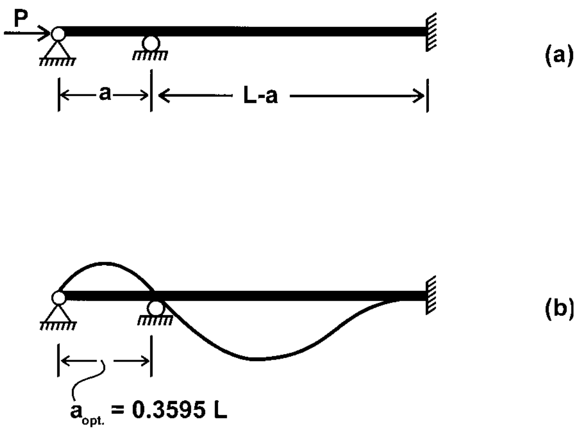

Fig. 4. (a) Uniform column simply supported at one end and clamped at the other end, with intermediate support. (b) Optimal placement of an intermediate support is the location of the node in the second bucking mode of a simply supported-clamped column without an intermediate support.

which reduces to:

$$-ak\omega_1 + a\omega_3 \sin ka + \omega_1 \sin^2\alpha + \omega_2 \sin\alpha\cos\alpha = 0 \tag{58}$$

Substituting the expressions for ω_j we get:

$$\begin{aligned} f(\alpha(u), u) &= (\sin\alpha u \cos\alpha u - \alpha u) \\ &\quad \times (\alpha\cos\alpha - \alpha u\cos\alpha + \sin\alpha u - \sin\alpha) \\ &\quad + \alpha u \sin\alpha u(\cos\alpha\cos\alpha u + \sin\alpha\sin\alpha u - 1) \\ &\quad + \sin^2\alpha u(\alpha\sin\alpha - \alpha u\sin\alpha \\ &\quad + \cos\alpha - \cos\alpha u) = 0 \end{aligned} \tag{59}$$

Before proceeding with the analysis of this equation we first concentrate on investigating the limiting behavior when $a \to 0$. Let us consider an expression for the slope in the second region, in the cross-section $x = a$:

$$\begin{aligned} w_2'(a) &= C(k\omega_1 \cos ka - k\omega_2 \sin ka + \omega_3) \\ &= C\{k\cos ka[(L-a)k\sin kL + \cos kL - \cos ka] \\ &\quad - kw_2 \sin ka + k[\cos k(L-a) - 1]\} \end{aligned} \tag{60}$$

In the limit, when a approaches zero, we get:

$$Lk^2 \sin kL + k\cos kL - k + k\cos kL - k = 0 \tag{61}$$

or,

$$Lk\sin kL + 2\cos kL - 2 = 0 \tag{62}$$

Using trigonometric identities:

$$\begin{aligned} \sin kL &= 2\sin\frac{kL}{2}\cos\frac{kL}{2} \\ 2\cos kL - 2 &= -4\sin^2\frac{kL}{2} \end{aligned} \tag{63}$$

we get,

$$2Lk\sin\frac{kL}{2}\cos\frac{kL}{2} - 4\sin^2\frac{kL}{2} = 0 \tag{64}$$

or, simply,

$$\tan\frac{kL}{2} = \frac{kL}{2} \tag{65}$$

which in the characteristic equation for the clamped-clamped column. Thus, when a tends to zero, the simply supported end at $x = 0$ and the intermediate support at $x = a$, tend to act in concert as a clamped end.

We return now to the characteristic equation (59). Recalling the notation of non-dimensional quantities u and α, as per equation (18), the characteristic equation is re-written as:

$$f(\alpha(u), u) = (u - u^2)\alpha\cos\alpha + 2\alpha u\sin\alpha u - \alpha u\sin\alpha + \sin\alpha u\cos\alpha u(\sin\alpha - \alpha\cos\alpha) - (\alpha\sin\alpha + \cos\alpha)\sin^2\alpha u = 0 \quad (66)$$

Let us check now the assertion, that the buckling load attains a maximum when the additional support is located at the node of the second buckling node. The characteristic equation for the buckling load of the column simply supported at one end and clamped at the other, and lacking an intermediate support, reads:

$$\tan\alpha = \alpha \quad (67)$$

The second root of this equation is:

$$\alpha = 7.7252 \quad (68)$$

Substituting equation (67) into equation (66), we get:

$$(\alpha\sin\alpha + \cos\alpha)\sin^2\alpha u - 2\alpha u\sin\alpha u + \alpha^2u^2\cos\alpha = 0 \quad (69)$$

Solving quadratic equation (69) with respect to $\sin\alpha u$, results in:

$$\sin\alpha u = \frac{2\alpha u \pm \sqrt{4\alpha^2u^2 - 4\alpha^2u^2\cos\alpha(\alpha\sin\alpha + \cos\alpha)}}{2(\alpha\sin\alpha + \cos\alpha)} \quad (70)$$

Due to equation (67) $\sin\alpha = \alpha\cos\alpha$, equation (70) reduces to:

$$\sin\alpha u = \frac{\alpha u}{(\alpha^2 + 1)\cos\alpha} \quad (71)$$

Substitution of the numerical value given in equation (68) yields:

$$\sin 7.725u = 0.9917u \quad (72)$$

The solution of equation (72) reads:

$$a = 0.3595L \quad (73)$$

which is the location of the node in the second buckling mode of the simply supported-clamped column without an intermediate support (Fig. 4b).

Having arrived at this result one can try to combine three cases of uniform columns considered above, namely (1) the column simply supported at its two ends, (2) the column clamped at its both ends, (3) the column simply supported at one end and clamped at the other. In the first two cases the maximum buckling load occurred if the additional support was placed in the middle cross-section of the column. This is a cross-section with respect to which the column exhibits symmetry. Yet, this location is also the node of the second buckling mode of the column without the intermediate support. Thus, all three cases appear to support the assertion that the buckling load takes a maximum value when the additional support is placed at the node of the second buckling mode of the column without the intermediate support. Yet, it appears still premature to claim that we have hit upon a right conclusion until the two remaining boundary condition cases are examined. These are the columns involving free ends.

6. UNIFORM COLUMN SIMPLY SUPPORTED AT ONE END AND FREE AT THE OTHER END, WITH INTERMEDIATE SUPPORT

The expression for the displacement in the first region, satisfying the conditions (Fig. 5a)

$$w_1(0) = w_1''(0) = w_1(a) = 0 \quad (74)$$

reads:

$$w_1(x) = A\left(\sin kx - \frac{x}{a}\sin ka\right) \quad (75)$$

The displacement in the second region, satisfying the conditions:

$$w_2(a) = w_2''(L) = w'''(L) + k^2w'(L) = 0 \quad (76)$$

is:

$$w_2(x) = C[\sin k(x - L) - \sin k(a - L)] \quad (77)$$

Now, let us calculate the derivatives:

$$\begin{aligned} w_1'(x) &= A\left(k\cos kx - \frac{1}{a}\sin ka\right) \\ w_1''(x) &= -Ak^2\sin kx \\ w_2'(x) &= Ck\cos k(x - L) \\ w_2''(x) &= -Ck^2\sin k(x - L) \end{aligned} \quad (78)$$

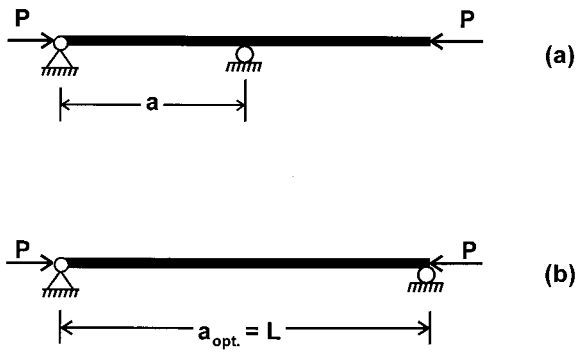

Fig. 5. (a) Uniform column simply supported at one end and free at the other, with intermediate support. (b) In order to maximize the buckling load one should place an additional support at column's free end.

Imposing the continuity conditions (9) and (10) yields:

$$A\left(k\cos ka - \frac{1}{a}\sin ka\right) - Ck\cos k(a-L) = 0 \quad (79)$$

$$-Ak^2\sin ka + Ck^2\sin k(a-L) = 0 \quad (80)$$

The following characteristic equation is obtained, from equations (79) and (80):

$$(ak\cos ka - \sin ka)\sin k(L-a) + ak\sin ka\cos k(L-a) = 0 \quad (81)$$

or with the familiar non-dimensional variables $u = a/L$, $\alpha = kL$, we get:

$$f(\alpha(u), u) = (\sin\alpha u - \alpha u\cos\alpha u)\sin\alpha(u-1) + \alpha u\sin\alpha u\cos\alpha(u-1) = 0 \quad (82)$$

Using the sum angle formula of trigonometry, equation (82) is reduced to:

$$f(\alpha(u), u) = \alpha u\sin\alpha + \sin\alpha u\sin\alpha(u-1) = 0 \quad (83)$$

Calculating the derivative of the left-side of equation (83) with respect to u, yields,

$$\alpha\sin\alpha + \alpha\cos\alpha u\sin\alpha(u-1) + \alpha\sin\alpha u\cos\alpha(u-1) = 0 \quad (84)$$

or

$$\sin\alpha = \sin\alpha(1-2u) \quad (85)$$

Solving for u in the latter equation we conclude that:

$$\alpha(1-2u) = \alpha \pm 2\pi m \quad (86)$$

where m is zero or an integer:

$$m = 0, \pm 1, \pm 2, \ldots \quad (87)$$

Thus

$$u = \frac{m\pi}{\alpha} \quad (88)$$

Substituting this value into the characteristic equation (83):

$$\pm m\pi\sin\alpha + \sin(m\pi)\sin\alpha\left(\pm\frac{m\pi}{\alpha} - 1\right) = 0 \quad (89)$$

or

$$\sin\alpha = 0 \quad (90)$$

The smallest non-trivial solution of equation (90) is:

$$\alpha = \pi \quad (91)$$

Bearing in mind equation (88), we get:

$$u = \pm m \quad (92)$$

where m is zero or an integer. On the other hand we observe, that through the definition of u in equation (18), it must lie in the closed interval [0, 1]. There are two solutions of equation (92) consistent with this requirement: $u = 0$ corresponds to $m = 0$, and $u = 1$ corresponding to $m = 1$. The condition $u = 0$ corresponds to the buckling load of the clamped-free column without an intermediate support with buckling load:

$$P_{cr} = \frac{\pi^2 EI}{4L^2} \quad (93)$$

The condition $u = 1$ corresponds to the buckling load of the column that is simply supported at its both ends, namely,

$$P_{cr} = \frac{\pi^2 EI}{L^2} \quad (94)$$

Thus, in order to maximize the buckling load of a simply supported-free column, one should place an additional support at column's free end (Fig. 5b).

Does the end of the free column correspond to the node of the second buckling mode of the simply supported free column without an additional support? In order to answer this question let us derive an expression for the buckling mode of such a column. The general solution for the displacement of the uniform column reads:

$$w(x) = B_1\sin kx + B_2\cos kx + B_3 x + B_4 \quad (95)$$

Satisfying the boundary condition $w''(0) = 0$ we conclude that $B_2 = 0$. The condition $w(0) = 0$ leads to $B_4 = 0$. Thus the displacement becomes:

$$w(x) = B_1\sin kx + B_3 x \quad (96)$$

Requirement $w''(L) = 0$ results in:

$$B_1 k^2\sin kL = 0 \quad (97)$$

From (97), since $B_1 \neq 0$, we require:

$$k = \frac{m\pi}{L}; \quad m = 0, \pm 1, \pm 2, \ldots \quad (98)$$

Using (96), the condition $w'''(L) + k^2 w'(L) = 0$, yields:

$$B_1 k^3(-\cos kL + \cos kL) + B_3 k^2 = 0$$

Hence, $B_3 = 0$. Thus, the normal modes are given by:

$$W_m(x) = B_1\sin\frac{m\pi}{L}x \quad (99)$$

which is the same as those of the column that is simply supported at both ends without intermediate support. The node of the second buckling mode ($m = 2$) occurs at $L/2$. Since we have demonstrated that the additional support which maximizes the buckling load should be placed at $x = L$, we conclude that the conjecture that the buckling load assumes a maximum value when the additional support is placed at the node of the second buckling mode of the column without intermediate support, is violated in the simple support-free case. Let us check the remaining case to be able to reach the final conclusions.

7. UNIFORM COLUMN CLAMPED AT ONE END AND FREE AT THE OTHER END, WITH INTERMEDIATE SUPPORT

In the first region the column's displacement satisfying the boundary conditions (Fig. 6a):

$$w_1(0) = w_1'(0) = w_1(a) = 0 \tag{100}$$

reads

$$w_1(x) = A[(\cos kx - 1)(\sin ka - ka) - (\cos ka - 1)(\sin kx - kx)] \tag{101}$$

In the second region the solution satisfying the conditions:

$$w_2''(L) = w_2'''(L) + k^2 w(L) = w_2(a) = 0 \tag{102}$$

reads

$$w_2(x) = C[\sin k(x-L) - \sin k(a-L)] \tag{103}$$

Satisfying the continuity conditions $w_1'(a) = w_2'(a)$; $w_1''(a) = w_2''(a)$ yields:

$$A[(-\sin ka)(\sin ka - ka) - (\cos ka - 1)^2] - C\cos k(a-L) = 0 \tag{104}$$

$$A[(-\cos ka)(\sin ka - ka) + (\cos ka - 1)\sin ka] + C\sin k(a-L) = 0 \tag{105}$$

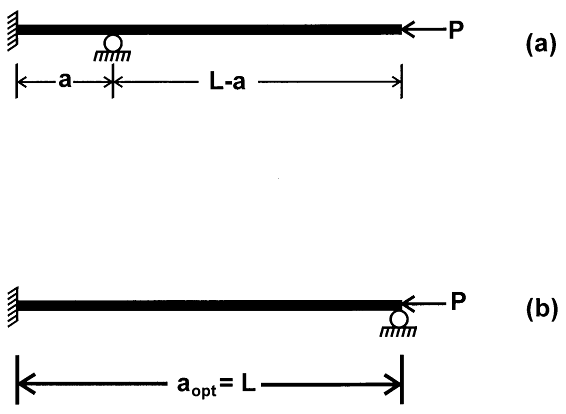

Fig. 6. (a) Uniform column clamped at one end and free at the other end, with intermediate support. (b) In order to maximize the buckling load, one should place an additional support at column's free end.

Requiring non-triviality of $A^2 + C^2$ we arrive at the characteristic equation:

$$\sin k(a-L)[-\sin ka(\sin ka - ka) - (\cos ka - 1)^2] + \cos k(a-L)[-\cos ka(\sin ka - ka) + \sin ka(\cos ka - 1)] = 0 \tag{106}$$

Using some trigonometric identities and introducing non-dimensional quantities α and u, as per equation (18), equation (106) is reduced to:

$$f(\alpha(u), u) = (-2 + \cos\alpha u)\sin\alpha(u-1) - \sin\alpha + \alpha u\cos\alpha = 0 \tag{107}$$

Taking a partial derivative of the left-hand side of equation (107) with respect to u yields:

$$-\sin\alpha u \sin\alpha(u-1) + (-2 + \cos ku)\cos k(u-1) + \cos\alpha = 0 \tag{108}$$

Let us set temporarily:

$$y = \alpha(u-1) \tag{109}$$

so that equation (108) can be re-written as follows:

$$-\sin(y+\alpha)\sin y + [-2 + \cos(y+\alpha)]\cos y + \cos\alpha = 0 \tag{110}$$

Expanding, using the trigonometric formulas for functions of sums and differences of two angles and collating like terms results in a quadratic equation for $\cos y$:

$$\cos^2 y - 2\cos\alpha\cos y + \cos^2\alpha = 0 \tag{111}$$

which could be rewritten as:

$$(\cos y - \cos\alpha)^2 = 0 \tag{112}$$

or, simply,

$$\cos y = \cos\alpha \tag{113}$$

Hence, on the one hand,

$$y = \alpha + 2m\pi, \quad m = 0, \pm 1, \pm 2, \ldots \tag{114}$$

and on the other, in accordance to the definition in equation (109):

$$y = \alpha(u-1) \tag{115}$$

Equating right sides of equations (114) and (115) we arrive at:

$$u = 2\left(1 + \frac{m\pi}{2}\right) \tag{116}$$

Since we require:

$$0 \le u \le 1 \tag{117}$$

we immediately observe from equation (116) that u

exceeds unity for all non-negative values of m. Let us, therefore, concentrate on the negative values of m. From equation (116):

$$\alpha u = 2\alpha + 2m\pi \tag{118}$$

We substitute this expression into the characteristic equation (107):

$$-3\sin\alpha + \cos 2\alpha \sin\alpha + (2\alpha + 2m\pi)\cos\alpha = 0 \tag{119}$$

The solution of this equation is:

$$\alpha = |m|\pi \tag{120}$$

Note that, the absolute value in equation (120) appears since the values of m under consideration are negative. Substituting this value into equation (118) we obtain $u = 0$.

We conclude from the above analysis that there is no relative maximum or minimum in the open interval $x \in (0, 1)$. However, since the function f, identified with the left side of equation (107) is continuous on a closed interval $[0, 1]$, it must obtain both a maximum value and a minimum value on $[0, 1]$. A simple check shows that the minimum occurs when $u = 0$ and the maximum is attained when $u = 1$.

Substituting this value into equation (107), we obtain:

$$-\sin\alpha + \alpha\cos\alpha = 0 \tag{121}$$

or

$$\tan\alpha = \alpha \tag{122}$$

which is the equation associated with the clamped-simply supported column.

Carrying out the corresponding calculation as was done in the simply supported free case without additional support, the modes for the clamped-free case without additional support are given by:

$$w_m(x) = B_1\left\{1 - \cos\left[\frac{(2m-1)}{2}\pi\frac{x}{L}\right]\right\} \tag{123}$$

It is seen that there are no nodes in the second buckling mode ($m = 2$) in the interval $(0, L)$ note that the third buckling mode ($m = 3$) has a node at $x = 4L/5$.

8. CONCLUSION

As is clearly seen in the two last cases the conjecture made by us that the buckling load takes a maximum value when the additional support is placed at the node of the second buckling mode of the column without the intermediate support, does not hold. In the last two cases the additional support must be placed at the free end of the column. Is it possible to put all obtained results as a single statement? The answer is affirmative. In order to maximize the buckling load of a uniform column by introducing a single support, it must be placed at the location where the first buckling mode of the column without intermediate support achieves a maximum. Indeed, if the displacement at both ends of the column vanishes, then the additional support must be placed at the location where the node of the second buckling mode of the column with the intermediate support removed is located. But this is also a location of the maximum displacement of the first buckling mode.

When one of the ends of the column is free, the additional support must be placed at the free end of the column. This is also a location of the maximum value attained by the first buckling mode of the column without support.

Thus all the cases fall in the same, unified category:

> In order to maximize the buckling load of a uniform column with classical boundary conditions at the ends, one first should locate the cross-section where the first buckling mode attains a maximum value; one then should place the additional support in that location.

In the actual course the conducted discussions on this topic were lively. We hope that analogous interest arise in many classes, when teaching the strength of materials course. The course should not be reduced to listing of results concisely and quickly covering them; material will be better digested if students participate in the discussions and are partially elevated to the status of co-uncoverers, maybe of even small exciting points.

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APPENDIX A

Derivation of equation (38)

The general solution for $w_1(x)$ is given by equation (11). The condition $w_1(0) = 0$ yields $A_4 = -A_2$. We calculate the slope:

$$w_1'(x) = A_1 k \cos kx - A_2 k \sin kx + A_3 \tag{A1}$$

The condition $w_1'(0) = 0$ results in:

$$A_3 = -A_1 k \tag{A2}$$

Thus, expression for the displacement becomes:

$$w_1(x) = A_1(\sin kx - kx) + A_2(\cos kx - 1) \tag{A3}$$

Imposition of the condition $w_1(a) = 0$ yields:

$$A_1(\sin ka - ka) + A_2(\cos ka - 1) = 0 \tag{A4}$$

or

$$A_2 = -A_1 \frac{\sin ka - ka}{\cos ka - 1} \tag{A5}$$

Substituting for A_2 results in:

$$w_1(x) = A[(\cos ka - 1)(kx - \sin kx) + (\cos kx - 1)(\sin ka - ka)] \tag{A6}$$

with A being a new constant:

$$A = A_1/(\cos ka - 1) \tag{A7}$$

Multiplying out in equation (A6) and using the identity:

$$(\sin kx)(\cos ka) - (\cos kx)(\sin ka) = \sin k(x-a) \tag{A8}$$

we obtain:

$$w_1(x) = A[-\sin k(x-a) - \sin ka - ka\cos kx + kx\cos ka + \sin kx - k(x-a)] \tag{A9}$$

APPENDIX B

Derivation of equation (39)

The general solution for $w_2(x)$ is given by equation (13). The imposition of the condition $w_2(L) = 0$ yields:

$$B_1 \sin kL + B_2 \cos kL + B_3 L + B_4 = 0 \tag{B1}$$

Now, the expression for the slope reads:

$$w_2'(x) = B_1 k \cos kx - B_2 k \sin kx + B_3 \tag{B2}$$

The condition $w_2'(L) = 0$ results in:

$$B_1 k \cos kL - B_2 k \sin kL + B_3 = 0 \tag{B3}$$

The condition $w_2(a) = 0$ yields:

$$B_1 \sin ka + B_2 \cos ka + B_3 a + B_4 = 0 \tag{B4}$$

We have from equations (B2), (B3) and (B4):

$$\begin{aligned} B_2 \cos kL + B_3 L + B_4 &= -B_1 \sin kL \\ -B_2 k \sin kL + B_3 + B_4 \cdot 0 &= -B_1 k \cos kL \\ B_2 \cos ka + B_3 a + B_4 &= -B_1 \sin ka \end{aligned} \tag{B5}$$

Solving these equations using Cramer's rule, we get:

$$\begin{aligned} B_2 &= B_1 \varphi_2/\varphi_1 \\ B_3 &= B_1 \varphi_3/\varphi_1 \\ B_4 &= B_1 \varphi_4/\varphi_1 \end{aligned} \tag{B6}$$

where

$$\begin{aligned} \varphi_1 &= \begin{vmatrix} \cos kL & L & 1 \\ -k\sin kL & 1 & 0 \\ \cos ka & a & 1 \end{vmatrix} \\ \varphi_2 &= \begin{vmatrix} -\sin kL & L & 1 \\ -k\cos kL & 1 & 0 \\ -\sin ka & a & 1 \end{vmatrix} \\ \varphi_3 &= \begin{vmatrix} \cos kL & -\sin kL & 1 \\ -k\sin kL & -k\cos kL & 0 \\ \cos ka & -\sin ka & 1 \end{vmatrix} \\ \varphi_4 &= \begin{vmatrix} \cos kL & L & -\sin kL \\ -k\sin kL & 1 & -k\cos kL \\ \cos ka & a & -\sin ka \end{vmatrix} \end{aligned} \tag{B7}$$

Substituting for B_2, B_3 and B_4 in terms of B_1 (equation B6), simplifying and introducing a new constant we obtain equation (39). As a check, by directly substituting the appropriate values for x in equation (39) it is immediately verified that equation (39) satisfies the conditions $w_2(L) = w_2'(L) = w_2(a) = 0$.

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