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On the Non-uniqueness of Characteristic Vectors*

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This paper provides two rules that describe the freedom available in the choice of the characteristic vectors of a system with multiple eigenvalues. The rules will be of help to students meeting the Jordan canonical form for the first time. The paper assumes a knowledge of the Jordan canonical form.

AUTHOR QUESTIONNAIRE

1. The paper discusses materials for a course in Modern Control Systems.
2. Students of the following departments are taught in this course: Mechanical Engineering, Manufacturing Engineering Systems.
3. Level of the course (year) is Level 3.
4. Mode of presentation includes lectures, seminars, and *Matlab* exercises. Material in the paper is presented within a lecture.
5. Hours required to cover the material is one hour. Student homework or revision hours required for the materials is one hour.
6. The novel aspects presented in this paper include two rules that describe the freedom available in the choice of the characteristic vectors of a system with multiple eigenvalues. The educational thinking behind the paper is to make students taking the course aware of the nature of the non-uniqueness of the characteristic vectors. Only the brightest students would be expected to seek an understanding of the lemmas that underpin the two rules.
7. The standard text recommended in the course, in addition to author's notes is K. R. Dutton, S. Thompson and B. Barraclough, *The Art of Control Engineering,* Addison-Wesley (1997). The material is not covered in the recommended text.

INTRODUCTION

IN Askarpour and Owens [1] the benefits of introducing students of Control Systems to the Jordan canonical form were summarised. In particular, Gajic and Lelic [2] use the Jordan canonical from to determine the stability of systems with multiple eigenvalues. Systems with multiple eigenvalues can be transformed into the Jordan canonical form by a similarity transformation, V, obtained by using characteristic vectors of the system. The characteristic vectors are the eigenvectors and generalised eigenvectors of the system. A simple unambiguous method for determining characteristic vectors of a system with multiple eigenvalues was presented in Askarpour and Owens [1]. However, the nature of characteristic vectors is often a source of confusion. In this paper, the non-uniqueness of the characteristic vectors of the Jordan canonical form is investigated.

It is easily seen that only some linear combinations of characteristic vectors can be employed to determine the Jordan canonical form leading to limited freedom in possible operations that can be performed on the transformation matrix without altering the shape of the Jordan form. For instance, simple examples of transformations to Jordan canonical form are provided by DeRusso *et al.* [3], and Borrelli and Coleman [4], and it is straightforward to check that the generalised eigenvectors used in these examples are not unique up to scalar multiples. DeRusso *et al.* [3] and Borrelli and Coleman [4] also provide useful introductions to specific issues involving deficient eigenspaces, generalised eigenvectors, and the Jordan canonical form. Operations on Jordan blocks are discussed by Gantmacher [5]. However, these operations are performed only on complete Jordan blocks and no proofs are given that the operations do not change the shape of the Jordan form.

In this paper, six lemmas are presented relating to which linear combinations of characteristic vectors can be employed to achieve the Jordan canonical form. From these lemmas two rules are deduced concerning the freedom available in the choice of the transformation matrix for the Jordan canonical form. The rules will be of help to students meeting the Jordan canonical form for the first time.

The lemmas proved in this paper are stated without proof in Askarpour and Owens [6].

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STATEMENT OF THE PROBLEM

Square matrices of dimension n which have repeated eigenvalues may be transformed into the Jordan canonical form using a similarity transformation. That is there exits a transformation V, obtained by using the characteristic vectors of the original matrix, such that:

$$J = V^{-1}AV$$

where J has the Jordan form.

Construction of a matrix V is achieved by searching for linearly independent eigenvectors from the standard eigenvalue-eigenvector problem

$$(A - I\lambda_i)v_i = 0$$

which produces $v \leq n$ linearly independent vectors and completing the n dimensional basis by using generalised eigenvectors of Jordan chains. To clearly express the chaining process the following notation is introduced:

- Let $v^l_{x,y,z}$ be an eigenvector or generalised eigenvector relating to eigenvalue λ_l where the subscript x shows the level of the vector in the chaining process e.g. $v^l_{0,y,z}$ denotes an eigenvector and $v^l_{2,y,z}$ denotes a second chained generalised eigenvector.
- The subscript y indicates the Jordan block with which the vector is associated.
- The maximum value that y takes is the geometric multiplicity of λ_l.
- The subscript z indicates the number of generalised eigenvectors chained off the vector, e.g. $v^l_{0,3,0}$ denotes the third eigenvector of λ_l which has no following chain and $v^l_{2,1,1}$ denotes the second chained generalised eigenvector of the first Jordan block which has one generalised eigenvector chained off it.

Now from the definition of eigenvectors and generalised eigenvectors we may write:

$$(A - I\lambda_l)v^l_{0,y,z} = 0$$
$$(A - I\lambda_l)v^l_{x+1,y,z-x-1} = v^l_{x,y,z-x} \quad (1)$$

and

$$(A - I\lambda_l)^{x+1}v^l_{x,y,z-x} = 0$$
$$(A - I\lambda_l)^{x}v^l_{x,y,z-x} = v^l_{0,y,z}$$

From the above it is clear that the nullity of $(A - I\lambda_l)$ is equal to the number of eigenvectors belonging to λ_l and the difference between the nullity of $(A - I\lambda_l)^2$ and the nullity of $(A - I\lambda_l)$ is equal to the number of first chain generalised eigenvectors belonging to λ_l and so on. Thus, it is conceptually simple to determine the Jordan form. The determination of a transformation matrix for the Jordan form is, however, a challenging problem. In this paper, rules that enable many transformation matrices that achieve the Jordan form to be determined, once one-transformation matrix is known, are developed.

NON-UNIQUENESS OF THE CHARACTERISTIC VECTORS

It is well known that the Jordan canonical form is unique up to the size and the order of the Jordan blocks. The following six lemmas determine which combinations of characteristic vectors can be used in the transformation matrix without altering the Jordan form.

Lemma 1. *Any generalised eigenvector can only be chained off one characteristic vector.*

Proof. Consider a generalised eigenvector $v^l_{i+1,j,k}$ which belongs to the repeated eigenvalue λ_l and is chained off the characteristic vector $v^l_{i,j,k+1}$.

For any ambiguity to occur due to a generalised eigenvector being produced from more than one characteristic vector, for example for $v^l_{i+1,j,k}$ to be produced from $v^l_{x,y,z} \neq v^l_{i,j,k+1}$ as well as $v^l_{i,j,k+1}$, it would mean that:

$$Av^l_{i+1,j,k} = \lambda_l v^l_{i+1,j,k} + v^l_{i,j,k+1}$$
$$= \lambda_l v^l_{i,j,k+1} + v^l_{x,y,z}$$

Hence:

$$v^l_{x,y,z} = v^l_{i,j,k+1}$$

Since the above is not true any generalised eigenvector can only be chained off one characteristic vector. □

Lemma 2. *Any linear combination of an eigenvector of a repeated eigenvalue and another eigenvector associated with the same eigenvalue with a smaller or equal number of chained vectors can replace the eigenvector with the smaller or equal number of chained vectors without altering the Jordan form.*

Proof. Consider:

$$\overline{v}^l_{0,j,k} = (\alpha v^l_{0,p,g} + \beta v^l_{0,j,k}),\ g \geq k$$

and

$$(A - I\lambda_l)v^l_{1,j,k-1} = v^l_{0,j,k}$$
$$(A - I\lambda_l)\overline{v}^l_{1,j,k-1} = (\alpha v^l_{0,p,g} + \beta v^l_{0,j,k})$$

From (1):

$$(A - I\lambda_l)\overline{v}^l_{1,j,k-1} = \alpha(A - I\lambda_l)v^l_{1,p,g-1} + \beta(A - I\lambda_l)v^l_{1,j,k-1}$$
$$\overline{v}^l_{1,j,k-1} = \alpha v^l_{1,p,g-1} + \beta v^l_{1,j,k-1}$$

It can be seen that one possible solution for $\overline{v}^l_{h,j,k-h}$ is:

$$\overline{v}^l_{h,j,k-h} = \alpha v^l_{h,p,g-h} + \beta v^l_{h,j,k-h},\ 0 < h \leq k$$

Hence, where $g \geq k$, $(\alpha v^l_{0,p,g} + \beta v^l_{0,j,k})$ will generate a chain of length k as required. □

Lemma 3. *No linear combination of an eigenvector of a repeated eigenvalue and an eigenvector associated with the same repeated eigenvalue with a longer following chain can replace the eigenvector with a longer following chain without altering the Jordan form.*

Proof. Consider $v^l_{0,q,k}$ and $v^l_{0,p,k+w}$ where $w > 0$. By definition, any linear combination of the two eigenvectors is also an eigenvector:

$$\overline{v}^l_{0,p,k+w} = (\alpha v^l_{0,q,k} + \beta v^l_{0,p,k+w})$$

Now consider the $(k+1)th$ generalised eigenvector, where:

$$(A - I\lambda_l)v^l_{k+1,p,w-1} = v^l_{k,p,w}$$

From the proof of Lemma 2, if $\overline{v}^l_{k,p,w}$ exists then one solution is:

$$\overline{v}^l_{k,p,w} = \alpha v^l_{k,q,0} + \beta v^l_{k,p,w}$$

so that if $\overline{v}^l_{k+1,p,w-1}$ exists then:

$$(A - I\lambda_l)\overline{v}^l_{k+1,p,w-1} = \alpha v^l_{k,q,0} + \beta v^l_{k,p,w}$$

and from (1):

$$\begin{aligned}(A - I\lambda_l)\overline{v}^l_{k+1,p,w-1} &= \alpha v^l_{k,q,0} \\ &\quad + \beta(A - I\lambda_l)v^l_{k+1,p,w-1}\end{aligned}$$

Hence, for the above to hold:

$$(A - I\lambda_l)\frac{(\overline{v}^l_{k+1,p,w-1} - \beta v^l_{k+1,p,w-1})}{\alpha} = v^l_{k,q,0}$$

The above says that, if there exists a $(k+1)th$ order generalised eigenvector generated from $(\alpha v^l_{0,q,k} + \beta v^l_{0,p,k+w})$ then there will also exist a generalised eigenvector which can be generated from $v^l_{k,q,0}$. The non-existence of such a generalised eigenvector proves Lemma 3. □

Lemma 4. *Any multiple of an eigenvector of a repeated eigenvalue added to a generalised eigenvector associated with the same eigenvalue with a shorter or equal length of chain can replace the generalised eigenvector with the shorter or equal length of chain without altering the Jordan form.*

Proof. Consider the eigenvector $v^l_{0,p,k+w}$, $w \geq 0$, and the generalised eigenvector $v^l_{c,q,k}$. The vector replacing the eigenvector is:

$$\overline{v}^l_{c,q,k} = (\alpha v^l_{0,p,k+w} + v^l_{c,q,k})$$

From equation (1) and by definition of the eigenvector:

$$(A - I\lambda_l)v^l_{c,q,k} = v^l_{c-1,q,k+1}$$

$$(A - I\lambda_l)v^l_{0,p,k+w} = 0$$

Multiplying the second equation by α and adding it to the first equation gives:

$$(A - I\lambda_l)(\alpha v^l_{0,p,k+w} + v^l_{c,q,k}) = v^l_{c-1,q,k+1}$$

The above shows that $(\alpha v^l_{0,p,k+w} + v^l_{c,q,k})$ satisfies the condition for being chained off $v^l_{c-1,q,k+1}$. To show the existence of k following chains consider:

$$(A - I\lambda_l)\overline{v}^l_{c+1,q,k-1} = \overline{v}^l_{c,q,k}$$

Substituting for $\overline{v}^l_{c,q,k}$ gives:

$$(A - I\lambda_l)\overline{v}^l_{c+1,q,k-1} = (\alpha v^l_{0,p,k+w} + v^l_{c,q,k})$$

and from that:

$$\begin{aligned}(A - I\lambda_l)\overline{v}^l_{c+1,q,k-1} &= \alpha(A - I\lambda_l)v^l_{1,p,k+w-1} \\ &\quad + (A - I\lambda_l)v^l_{c+1,q,k-1}\end{aligned}$$

so that one solution for $\overline{v}^l_{c+1,q,k-1}$ is:

$$\overline{v}^l_{c+1,q,k-1} = \alpha v^l_{1,p,k+w-1} + v^l_{c+1,q,k-1}$$

It can be shown that:

$$\overline{v}^l_{c+h,q,k-h} = \alpha v^l_{h,p,k+w-h} + v^l_{c+h,q,k-h},\ 0 < h \leq k$$

From the above the existence of k chained vectors is deduced which proves Lemma 4. □

Lemma 5. *No linear combination of a generalised eigenvector of a repeated eigenvalue and an eigenvector associated with the same repeated eigenvalue can replace the eigenvector without altering the Jordan form.*

Proof. By definition:

$$(A - I\lambda_l)v^l_{0,q,w} = 0$$

$$(A - I\lambda_l)v^l_{c,p,k} = v^l_{c-1,p,k+1}$$

Multiplying the second equation by α and adding it to the first equation:

$$(A - I\lambda_l)(\alpha v^l_{c,p,k} + v^l_{0,q,w}) = \alpha v^l_{c-1,p,k+1}$$

Clearly, $(\alpha v^l_{c,p,k} + v^l_{0,q,w})$ does not satisfy the definition of an eigenvector. □

Lemma 6. *No linear combination of a generalised eigenvector of a repeated eigenvalue with a second generalised eigenvector associated with the same repeated eigenvalue can replace the second generalised eigenvector without altering the Jordan form.*

Proof. Consider a linear combination of two generalised eigenvectors:

$$(A - I\lambda_l)v^l_{d,q,w} = v^l_{d-1,q,w+1}$$
$$(A - I\lambda_l)v^l_{c,p,k} = v^l_{c-1,p,k+1}$$
$$(A - I\lambda_l)(\alpha v^l_{c,p,k} + \beta v^l_{d,q,w}) =$$
$$= \alpha v^l_{c-1,p,k+1} + \beta v^l_{d-1,q,w+1}$$

Clearly, the vector $(\alpha v^l_{c,p,k} + \beta v^l_{d,q,w})$ does not satisfy the definition of a generalised eigenvector chained off $v^l_{c-1,p,k+1}$. □

TWO RULES ON THE FREEDOM AVAILABLE

The results of the six lemmas proved above can be summarised in the following two rules:

Rule 1. *No linear combination of a generalised eigenvector of a repeated eigenvalue and any other characteristic vector associated with the same repeated eigenvalue can replace the latter without altering the Jordan form. This also means that a generalised eigenvector can not be replaced by multiples of itself.*

Rule 2. *A multiple of an eigenvector of a repeated eigenvalue can only be added to a generalised eigenvector or multiple of an eigenvector associated with the same repeated eigenvalue to replace the latter without altering the Jordan form if the length of the chain of the former is greater than or equal to the length of the chain of the latter.*

APPLICATION OF THE RULES

Consider the 7th-order matrix:

$$\mathbf{A} = \begin{bmatrix} -3 & 1 & 0 & 1 & 3 & 1 & 1 \\ -1 & 1 & -3 & -1 & 7 & 1 & 1 \\ 1 & -1 & -1 & 2 & -3 & -1 & -1 \\ 0 & 0 & 0 & -2 & 0 & 0 & 0 \\ 1 & -2 & 2 & 2 & -7 & -1 & -1 \\ -4 & 6 & -4 & -6 & 16 & 2 & 4 \\ 2 & -3 & 2 & 3 & -8 & -2 & -4 \end{bmatrix}$$

with one multiple eigenvalue at -2.

Application of the method of Askarpour and Owens [1] for finding the characteristic vectors of matrix with repeated eigenvalues gives one solution for the matrix of characteristic vectors as ([1]):

$$\mathbf{V}_1 = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & -1 & 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & -1 & 0 & 0 \\ 2 & 2 & -6 & 0 & 2 & 0 & 0 \\ -1 & -1 & 3 & 0 & -1 & 0 & 1 \end{bmatrix}$$

with cond $(\mathbf{V}_1) = 137.5668$

and $\mathbf{V}_1^{-1}\mathbf{A}\mathbf{V}_1 = \mathbf{J}$ where:

$$\mathbf{J} = \begin{bmatrix} -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 \end{bmatrix}.$$

Application of the rules for combining characteristic vectors allows for simple generation of other characteristic matrices without altering the Jordan form. For example, any linear combination of the three eigenvectors can replace $v_{0,3,0}$ and any linear combination of the eigenvectors and $v_{3,1,0}$ replace the latter. In particular, $v_{0,1,3}$ can be added to $v_{0,2,1}$ to replace the latter but, from the results of Lemma 2, to maintain the Jordan form $v_{1,1,2}$ must then be added to $v_{1,2,0}$ to replace the latter so another possibility for the characteristic matrix is:

$$\mathbf{V}_2 = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 2 & 2 \\ 0 & 1 & -1 & 2 & 2 & 2 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & -1 & -1 & 0 & 0 \\ 2 & 2 & -6 & 2 & 4 & 2 & 2 \\ -1 & -1 & 3 & -1 & -2 & -1 & 0 \end{bmatrix}$$

with cond $(\mathbf{V}_2) = 113.5422$.

Furthermore, $v_{0,2,1}$ can be added to $v_{2,1,1}$ to replace the latter but, from the results of Lemma 4, to maintain the Jordan form $v_{1,2,0}$ must then also be added to $v_{3,1,0}$ to replace the latter so another possibility for the characteristic matrix is:

$$\mathbf{V}_3 = \begin{bmatrix} 1 & 1 & 2 & 2 & 1 & 2 & 2 \\ 0 & 1 & 1 & 4 & 2 & 2 & 0 \\ 0 & 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & 0 & 0 \\ 2 & 2 & -2 & 4 & 4 & 2 & 2 \\ -1 & -1 & 1 & -2 & -2 & -1 & 0 \end{bmatrix}$$

with cond $(\mathbf{V}_3) = 84.8714$.

CONCLUSIONS

The freedom available in the choice of the transformation matrix for the Jordan canonical form has been given in terms of two rules formulated from six lemmas relating to which linear combinations of characteristic vectors can be employed to achieve the Jordan canonical form.

The topic of eigenvalues and eigenvectors is extremely important in Control Engineering as much of system analysis and synthesis depends upon the solution of eigenvalue problems. For completeness it is appropriate for students to consider the case of multiple eigenvalues. However, student texts which do consider the case of multiple eigenvalues do not address the non-uniqueness of the characteristic vectors of Jordan blocks. It is important for students to realise that generalised eigenvectors are, unlike the eigenvectors associated with distinct eigenvalues, not unique up to scalar multiples and that only some eigenvectors associated with repeated eigenvalues allow the chaining of generalised eigenvectors. The two rules presented are of educational value in so far as they highlight these issues. The two rules are also of practical value in that they can be used in conjunction with the algorithm of Askarpour and Owens [1] to construct a well-conditioned transformation matrix for the Jordan canonical form.

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