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Using the WWW in Multivariable Calculus to Enhance Visualization*

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As part of the reform movement in undergraduate mathematics education, instructors have been encouraged to take advantage of technology for teaching. At the University of Oklahoma, we chose multivariable calculus as the course in which to begin to increase the use of computers—particularly to facilitate visualization of the three-dimensional objects studied in that course. In particular, this paper advocates use of the World-Wide-Web to disseminate multivariable calculus materials to students and to instructors. We begin by presenting an abbreviated example from the materials we have developed; we then offer a variety of options for using such materials. Results from faculty and student surveys provide context and an indication of the need for more extensive use of the Web as a resource. We expect that, as appropriate materials are made available and promoted, use will increase and lead to enhanced student learning.

INTRODUCTION

FOR MORE THAN a decade there has been a movement to reform undergraduate mathematics, with the initial effort at the calculus level [1]. These activities clearly have a direct impact on engineering students, because so much of their work depends on a solid foundation in calculus. As part of the reform movement in undergraduate mathematics education, instructors have been encouraged to take advantage of technology for instruction [2]. Motivations for this effort are many and varied: the possibility of being able to emphasize conceptual learning rather than just computation, the ability to handle more complicated problems, and a closer connection with the ways that mathematics is used by the workforce. Although these issues are under debate, for the authors and others in the mathematics community the question has not been *whether* to implement technology but *how* to implement it in appropriate, effective ways.

With respect to technology, classroom instruction has ranged (nationally) from traditional lecture, to calculator-intensive explorations, and into computer-based curricula (for a partial listing of projects and resources see [3]). At the University of Oklahoma (OU), some sections of differential and integral calculus have made use of graphing calculators. However, these calculators typically have not been able to represent the three-dimensional objects studied in multivariable calculus. Thus, we chose multivariable calculus as the course with which to begin to increase the use of computers; two of six sections in Fall 1998 were the initial pilot classes and three of six sections in Spring 1999 were designated to participate in this project. In particular, we wanted to take advantage of the power of a computer algebra system (CAS) to facilitate the visualization of three-dimensional objects. (The OU Department of Mathematics has been using *Mathematica* instead of other software packages because OU already had a site license for *Mathematica*, so there was no additional cost to the students or to the institution.)

Having recognized that "increasing the use of computers" can involve a variety of possible activities, we chose to use computers in three ways:

1. We used a CAS to produce graphics and animations that were displayed during class time.
2. We assigned problems that introduced students to the basics of using the CAS for graphing and computing.
3. We created a website [4] that includes, among other things, the graphics displayed in class, with the intent that students from all sections—not just the sections designated to use the CAS—could make use of the resources.

Since one goal of the ongoing project has been to make available resources from which instructors can choose to incorporate computers more easily at a level they deem appropriate, this paper discusses possibilities for using the World-Wide-Web to disseminate materials to students and to other instructors. (A discussion of the use of the CAS with this project can be found in Murphy *et al.* [5].) We begin by presenting an abbreviated example of the materials available, then we offer a variety of options for using materials like these. We also include results from a faculty survey and a

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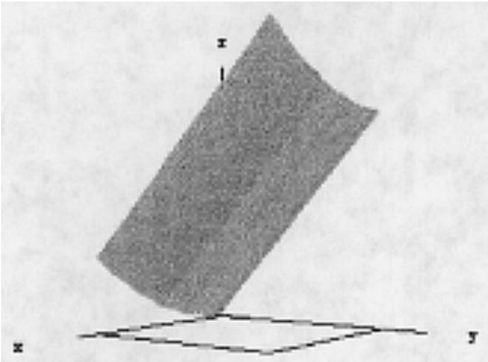

Fig. 1. The surface $f(x, y) = x^2 + 4y$ restricted to $D = \{(x, y)|0 \leq x \leq 2, 0 \leq y \leq 3\}$.

student survey to provide context and an indication of the need for projects like this.

AN EXAMPLE

One of the skills we seek to enhance is mental visualization of the three-dimensional objects studied in multivariable calculus. For this purpose, we created sets of overhead transparencies (and, in some cases, computer animations) that depict a mathematical process as a sequence of events. For some topics, textbook illustrations tend to be complicated—offering only the final diagram that includes all of the objects and labels. In this section, we offer a sample of a topic for which a sequence of slides (or, better yet, an animation) is preferable, rather than just the final picture.

In multivariable calculus students learn to set up Riemann sums to calculate the volume under a surface. We introduce this topic with a specific example and several *Mathematica*-generated graphics to illustrate the process. The example offered here (from Stewart [6]) uses the surface $f(x, y) = x^2 + 4y$ restricted to the rectangle $D = \{(x, y)|0 \leq x \leq 2, 0 \leq y \leq 3\}$, as pictured in Fig. 1.

We want to find the volume of the region that lies between this surface and the xy-plane. To help students get a feel for the shape of this object we offer several viewpoints (Fig. 2).

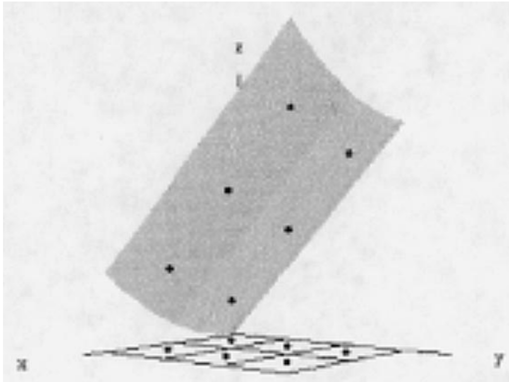

Fig. 3. A partition of the domain rectangle and associated sample points.

Since the process uses 'boxes' to estimate the volume under the surface, we show students visually how to construct the boxes. Figure 3 shows the partition of the domain rectangle D—the base rectangles for the boxes—the selection of points in each of the subrectangles, and the corresponding points on the surface that determine the height for each box.

Subsequently, Fig. 4 shows the construction of the boxes and offers several perspectives from which to view the result.

Typically textbooks offer only one illustration that includes the surface, the partition of the base rectangle, a sample point, the box edges, and notational labels for all of the parts of the diagram. We think that such busy illustrations include too much information for students to decode, particularly since students are not yet entirely comfortable trying to visualize the objects and processes. By separating the textbook illustration into a sequence of graphics, we can assist students' internal development of this concept.

THE 'CALCULUS AT OU' WEBSITE

In addition to facilitating student learning, another goal of the ongoing project has been to make available a variety of materials that would enable instructors, at OU or elsewhere, to increase the appropriate use of technology with their

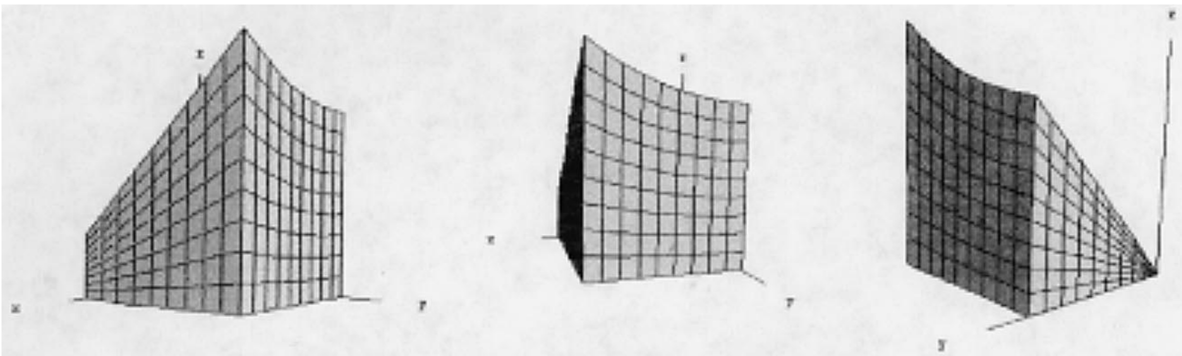

Fig. 2. The solid object that lies below the surface.

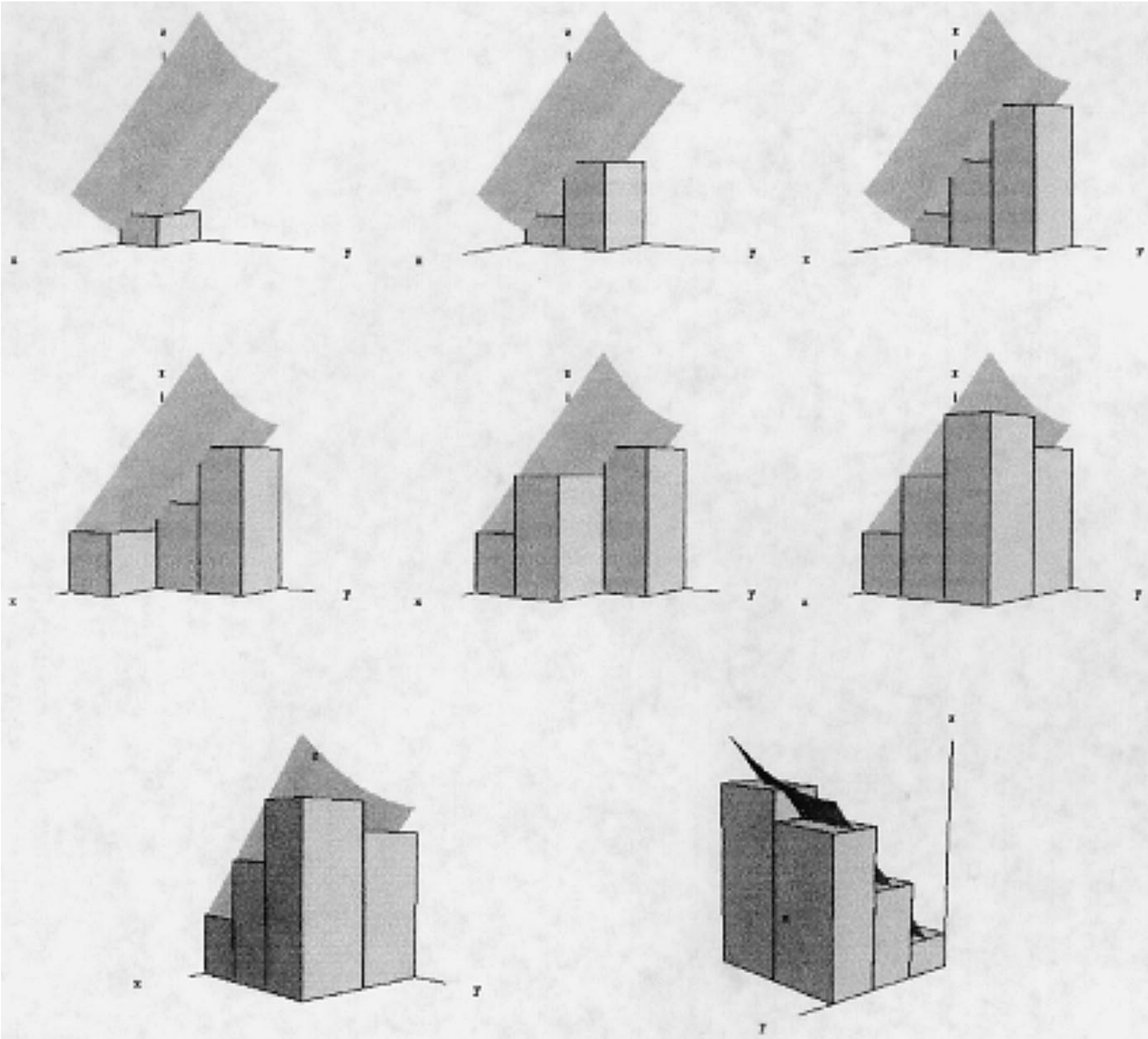

Fig. 4. Approximating the volume with boxes.

classes. To this end, the Department of Mathematics at OU has created a 'Calculus at OU' website [4]. The above sequence of graphics is one of several such sets available at the website. Other topics include:

- Construction of a contour diagram
- Construction of a tangent plane
- Directional derivatives and the gradient
- The geometry of Lagrange multipliers
- Double integrals
- Surface area
- Parametric surfaces: their tangent planes and surface areas.

Each topic is offered as an HTML page that includes the graphics (as GIFs), the *Mathematica* commands (as GIFs) used to generate the graphics, and explanatory text (much like the commentary text in the example above). Additional topics will be added as needed and as the project team's efforts move into other courses. To maximize access, all of these resources are available as HTML documents rather than as *Mathematica* documents, so that web users who do not have access to *Mathematica* can nevertheless take advantage of the graphics and text. Including the *Mathematica* commands offers the advantage that students and instructors can manipulate the commands to customize the examples—a feature of these resources that we hope can facilitate deeper understanding for students and add flexibility for instructors.

In a limited capacity (to be expanded over time), the website also houses example problem sets, solution keys, and reproductions of in-class demonstrations. A graduate student has been hired as webmaster to maintain, update, and develop the site.

SOME OPTIONS TO INCREASE THE USE OF COMPUTERS

We believe that instructors can use these resources to facilitate teaching while students can use them to reinforce their in-class experiences. In the first semester of the project, Fall 1998, two instructors of the multivariable calculus course at OU began to develop the materials discussed above. In Spring 1999, these two instructors continued to develop and use the materials: they used the previously prepared graphics; they also used the Web to transmit material to students; and they had their students work assignments that required the use of *Mathematica*. The prepared overhead transparencies were made available to other instructors in the form of an evolving 'instructor's manual'. As the materials were developed, a third instructor—not directly associated with the project—used some of the graphics in class as well. A fourth instructor said that he also used some of the overhead transparencies in a more limited way. The hope is that additional instructors will be able to make use of this resource as it is increasingly made available and promoted. This section describes some ways that we think these materials might be used.

The use that involves the least effort from an instructor is to 'advertise' the 'Calculus at OU' website to students and encourage them to look there for additional examples. Student use of the web page should have at least two benefits:

1. Some of the graphics—for example, Fig. 4—are actually presented as animated GIFs on the Web; these animations are intended to strengthen student understanding of and visualization of mathematical processes.
2. Students have the option of printing out the pages so that they can make notes on them and study the graphics, a strategy that some people find preferable to the use of 'course packets' or in-class handouts since students can select for themselves the topics that they want to print.

A second possible way that instructors can use the project materials is to display the graphics in class. Sets of overhead transparencies have been made available so that instructors can select the graphics they wish to display. One instructor at OU commented that she valued the time she saved by not having to draw the graphics by hand during class. Since the graphics are available at the website, students can complete their class notes by printing out the appropriate web pages. This might be particularly valuable for instructors who are teaching the course for the first time and thus have not yet developed complete sets of their own materials. This option is also available to instructors outside of OU (with only marginally more effort) since the graphics are available on the web.

Beyond the use of the pre-prepared overhead transparencies, an instructor has the option of presenting graphics as in-class demonstrations. This, of course, requires demonstration equipment (perhaps a laptop and a projection device), but also takes full advantage of the animations. An instructor can present the demonstrations using the *Mathematica* commands or simply by using the website. Again this option is available to instructors both at OU and elsewhere by way of the website.

With some effort instructors can write their own examples or create a different sequencing of graphics. Each selected topic includes, along with the graphics, the *Mathematica* commands used to generate the graphics. In addition, an undergraduate student has written a *Mathematica* package to facilitate the use of additional or alternate examples (since this effort is still under way, this package currently is only available from the first author by request). Thus instructors can modify the examples to fit their own styles and lesson plans.

Finally, the most extreme use of these materials is to have students use *Mathematica* themselves. Starting from the *Mathematica* commands available at the website as prototypes, students can modify the commands to fit the specific problem they need to solve. Doing this for all material in a multivariable calculus class is a considerable undertaking, but for selected topics it might facilitate understanding of larger concepts without undue distraction from computational details.

EVIDENCE OF NEED

As part of an effort to assess our activities, we administered a survey to students in the six sections of multivariable calculus in Spring 1999. The response rate was: 81 out of 103 (79%) students in the three sections designated to participate in the project ('project sections') and 66 out of 86 (77%) students in the sections not designated to participate in the project ('non-project sections'). The survey included the following items related to the use of computers.

For my multivariable calculus class I:

- See graphics that were generated by a computer (on TV monitor or overhead projector).
- Use a computer algebra system to generate graphics (in-class or out-of-class).
- Do computations with a computer algebra system (in-class or out-of-class).
- use the web (class website, other websites, downloadable software, etc.).

These response options were provided:

1. Not available.
2. Available, not used.
3. Used less than once a month.
4. Used at least once a month.
5. Used at least once a week.

We used an exploratory data analysis procedure called 'median polishing' [7] to examine the responses. This analysis drew attention to differences that we then verified using ANOVA (with an alpha level of 0.05 for all statistical tests unless otherwise indicated).

Figure 5 highlights these differences, aggregating the categories 'used at least once a month'

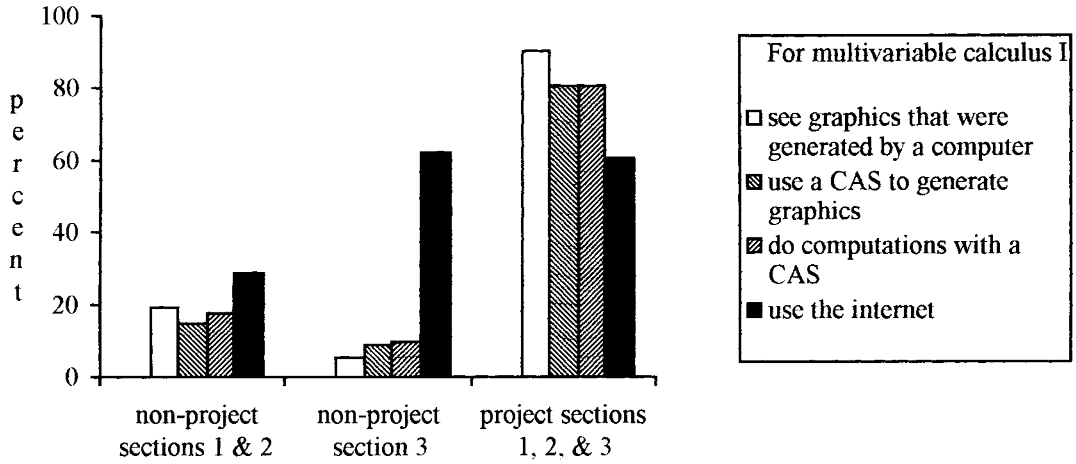

Fig. 5. Percent of students who responded 'used at least once a month' or 'used at least once a week'.

and 'used at least once a week' to simplify the display. In particular we found that the project section means were not significantly different for any of the four items, hence we aggregated these three sections in Fig. 5 (respectively for the four items: $F(2,78) = 1.84$, $p = 0.17$; $F(2,78) = 1.74$, $p = 0.18$; $F(2,78) = 1.88, p = 0.16$; and $F(2,78) = 1.73$, $p = 0.18$). On the other hand, the three non-project sections did not have different means on three items (respectively for the three items: $F(2,63) = 1.98$, $p = 0.15$, $F(2,63) = 0.004$, $p = 0.996$; and $F(2,63) = 0.34$, $p = 0.71$), but one of the non-project sections had its mean significantly different from the other two sections on use of the Internet ($F(2,63) = 3.60, p = 0.03$; using the Tukey method of multiple comparisons yielded $q(63,3) = 3.29$ and $q(63,3) = 3.24$ with $p < 0.10$ in both cases). Thus, Fig. 5 has the responses aggregated for two of these three sections, but the third section is separated.

The results emphasize that the two instructors of the three sections involved in the project did indeed make use of computers, with 48–96% of the students responding 'used at least once a month' or 'used at least once a week' for all four items. In contrast, students in two of the three non-project sections made clear that computers were not used much, if at all, for any of the options listed in the items. In the remaining non-project section, 62% of the students indicated that they sometimes used the Internet, but not a CAS, for their multivariable calculus class.

The survey also included the following item: Indicate how helpful each of the following has been for you in learning multivariable calculus material:

- Computer Algebra System (e.g., *Mathematica*, *MathCad*).
- Web (class website, other websites, downloadable software, etc.).

These were the possible responses:

1. Does not apply.
2. Not helpful.
3. Somewhat helpful.
4. Very helpful.

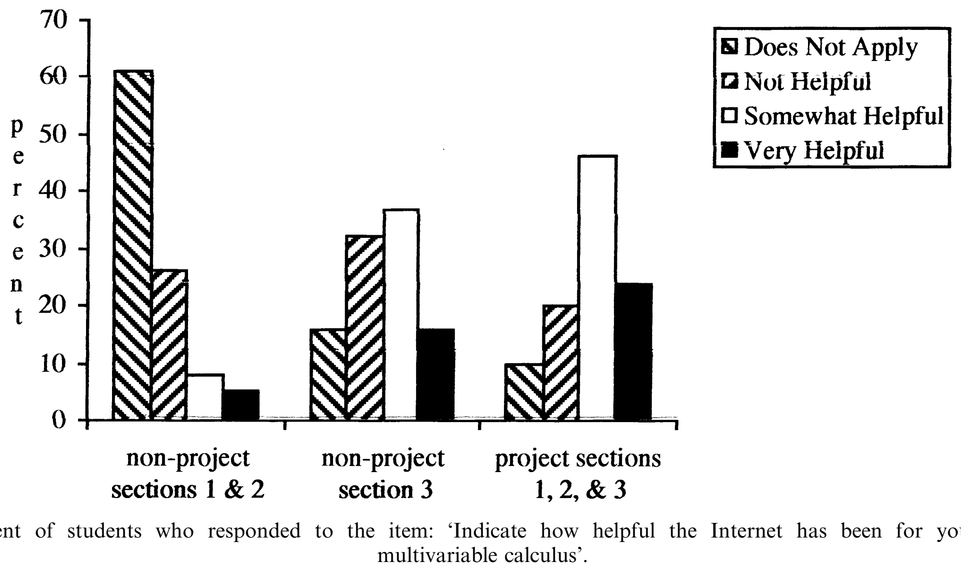

Fig. 6. Percent of students who responded to the item: 'Indicate how helpful the Internet has been for you in learning multivariable calculus'.

In response to the first part, 79% of the students in the three non-project sections indicated that a CAS did 'not apply' or was 'not helpful'—not surprisingly since they previously indicated that such technology was not being used for their calculus class. On the other hand, 94% of the students in the project sections responded that a computer algebra system had been 'somewhat helpful' or 'very helpful'—suggesting that, despite the extra effort required to learn to use the CAS, students who used it believed it was helpful.

Responses to the item addressing helpfulness of the web were more complicated to analyze since students in one of the non-project sections indicated some use of the Web. Figure 6 depicts the distribution of student responses, combining the non-project sections in which students indicated little use of the web, and combining the three project sections. As expected, students who indicated non-use did not find the web helpful. Those students who indicated at least some use of the web found it at least somewhat helpful—regardless of project participation status. To us this result indicates that students want all the help they can get.

The questions on the student survey did not refer specifically to the 'Calculus at OU' website because it was not ready for mass-promotion (for example, we did not receive funding for a webmaster position until we were well into the spring semester).

To make the website more effective and accessible, near-future plans for the 'Calculus at OU' website include:

1. Institutionalizing the webmaster position so that it is permanently funded.
2. Expanding the collection of materials at the website to include resources for the entire calculus sequence (such as an electronic reserve for instructors' course materials).

One influence on both of these goals is the perceived need in the department; that is, the faculty's desire to take advantage of the calculus website to house information for students throughout the calculus sequence.

To gather information about the perceived need, we distributed a survey to 25 permanent mathematics faculty. Of the 11 respondents, six indicated that they already make use of the Web for their classes in a variety of ways (not necessarily the 'Calculus at OU' website), providing course information and content as well as links to other pertinent websites. The remaining five indicated that they have considered or would be interested in using the Web as a tool for teaching. All five of these latter respondents indicated that they have not used the Web because it is 'too time consuming' or they have 'little knowledge of the Web or HTML'. In addition we asked, 'If a service were made available to assist in your teaching or class administration by means of the Web, would you be interested in taking advantage of it?' Each of the eight who responded to this item said 'Yes.' These results indicate to us that there is interest in taking advantage of the Web as a teaching tool, but faculty are reluctant to invest their time in this activity. We hope to bridge the gap between faculty interest in using the Web and their ability to do so. For example, the (graduate student) webmaster will be responsible for making instructors' materials accessible through the site or facilitating instructors' efforts to do so. Such a scenario illustrates one way that a permanent webmaster can help the balance between the need for web-based course materials and faculty resistance to this time-consuming task.

CONCLUSION

In the six sections of multivariable calculus taught at OU in Spring 1999, 66% of the students who responded to the survey were engineering majors. (In this case, the category 'engineering majors' does not include those students majoring in computer science.) This population of students at OU is increasingly gaining experience with computers by way of engineering and support courses. (In fact, at OU, the College of Engineering has recently implemented a program through which incoming students are expected to purchase a laptop computer.) Calculus, which occurs so early in the engineering curriculum, is an obvious place to work at increasing student comfort level with computers. In particular, we used a computer algebra system to develop materials intended for deepening understanding and enhancing their visualization skills.

We made our materials available to students (in a limited way) via the World-Wide-Web. In the three project sections and one of the non-project sections, students responded that they used the Web at least sometimes and that they found it at least somewhat helpful. Even in the two remaining non-project sections, students responded the most positively to items about the WWW. Faculty have also indicated a willingness to make use of the Web as a resource. As the project evolved, other instructors expressed an interest in our materials. A more formal survey of the permanent mathematics faculty at OU confirmed interest in use of the web and offered some idea of the scope of the need for added convenience. We expect that as appropriate materials are made available and promoted—especially to instructors of and students in non-project sections—use will increase and so will perception of value. We also expect that increases in use and positive attitude will contribute to student learning.

As indicated by faculty and student survey results, academia needs to expedite use of the Web as a teaching tool. In this paper we offer a variety of ways that projects such as this one can use the Web to facilitate the use of computers in teaching (not just in mathematics courses!) making materials available to a broad audience who can use the resources in ways that they feel are comfortable and appropriate. The WWW is a natural—but underutilized—resource, accessible in ways that instructors, classmates, colleagues, and library materials are not.

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