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Learning Differential Equations by Exploring Earthquake Induced Structural Vibrations: A Case Study*

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This paper discusses an interdisciplinary project administered at the US Military Academy in a calculus with differential equations course. The project takes full advantage of analytical, numerical, and qualitative techniques to explore the mechanical vibrations of buildings during an earthquake. The traditional treatment of mechanical vibration topics in an elementary ODE course is extended through the use of non-dimensionalization, energy related concepts, Coulomb damping, systems of differential equations and the use of computer animations to develop physical intuition and engineering concepts. A primary goal is to allow students to experience both the capabilities and the limitations of technology.

INTRODUCTION

MECHANICAL AND ELECTRICAL vibrations are routinely covered in courses on ordinary differential equations (ODE's) because they exemplify mathematical models of the real world [1, 2, 5, 7, 11, 12]. The differential equations are simple, yet the applications are extremely important to physics and engineering. The project described in this paper is an extension of an interdisciplinary project administered at the US Military Academy (USMA) last year in an undergraduate Calculus with Differential Equations course. It was designed in response to USMA's Civil and Mechanical Engineering Department's desire for students to have a better understanding of vibrations and systems of differential equations.

The project invites students to explore the earthquake-induced mechanical vibrations of buildings. Students discover both standard and nonstandard vibration topics while applying concepts in a realistic setting. The project takes full advantage of analytic, numerical, and qualitative techniques to address a large range of mathematical and engineering concepts. Throughout, student familiarity with a computer algebra system (CAS) is assumed. To prevent students from using technology as a crutch instead of as a tool, a primary goal of the project is for students to determine the **appropriate** use of the available technology. In some cases, standard numerical ODE solvers do not perform well without considerable tinkering. In others, analytical solutions cannot be found completely without use of physical intuition to address modeling assumptions and numerical limitations of a CAS. Also, the algebra is sometimes overwhelming, and it is difficult to proceed by hand. Finally, CAS's provide a simple means of animating the dynamical system—it is easy to actually see a model of the building vibrate!

The standard mechanical vibration problem consists of a block of mass m attached to a mounted spring of negligible mass with stiffness k. One readily derives the following equation of motion using Hooke's law and Newton's second law of motion:

$$mu'' + \gamma u' + ku = F(t) \tag{1}$$

where u is the displacement of the block from equilibrium, $\gamma u'$ represents viscous damping, and $F(t)$ is a time-dependent applied external force. This equation is solved by methods traditionally covered in an undergraduate ODE class. Seeking solutions of the form $u = e^{rt}$ leads to the characteristic equation:

$$mr^2 + \gamma r + k = 0. \tag{2}$$

The qualitative behavior of the solutions depends on the sign of the discriminant $\gamma^2 - 4mk$. Particular solutions can be found by standard methods, and for sinusoidal forcing functions, beat and resonance phenomena are observed if the forcing function is appropriately tuned. In this paper, we apply the same ODE to model the vibrations of a single-story building. The project includes the

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traditional topics described above, but extends the study in four major ways. First, we non-dimensionalize to remove explicit dependence on m and k to reduce the number of physical constants involved. This makes the equations more tractable without any loss of generality. Second, we examine a model with Coulomb damping, which is a damping mechanism that yields a *nonlinear* ODE which can still be solved by elementary means. Solutions tend to *nontrivial* equilibria in *finite* time. Third, we introduce the use of energy concepts to interpret the physical behavior of the models investigated. Fourth, we consider systems of ODE's to model buildings with multiple floors. Many of these topics are now reasonable for students because of the availability of compute algebra systems.

THE PROJECT

Our goal is to analyze the effects of an earthquake on a building. Although it is impossible to completely protect a building from the effects of an earthquake, it is possible to effect building designs which minimize the damage that a building is likely to sustain. In this project, we model the lateral motion of a building, and examine what can cause excessive displacements of a given floor, leading to significant damage of a building. One of the novelties of the project is that it was written as a self-exploration exercise. The students were shown how to model a spring-mass system for the simple harmonic case. They were expected to derive, on their own, other models which included the two types of damping, forcing functions, and multiple stories. Additionally, at the time the project was administered, the students were working with the method of undetermined coefficients. As the reader shall see, assigning this project at this time motivates other subsequent topics to be discussed in a typical ODE course.

We emphasize that students were free to solve problems analytically or numerically. In this way, they encountered strengths and weakness of both approaches, and learned how to use technology wisely.

As an introduction, we begin by modeling the vibrations of a single story building. After this is mastered, the behavior of multi-story buildings is considered. In either case, catastrophe strikes when a floor of the building experiences excessive lateral deflection, causing permanent damage to the structure. It is too difficult to model the fine details of the motion of a building. Instead, we model buildings as idealized structures consisting of relatively heavy, inextensional floors and light, elastic walls. Considering each floor of the building as a point mass located at the center of mass of the floor (Fig. 1), the analogy with a spring/mass/damper system (Fig. 2) is clear. The walls provide elastic forces which act opposite the direction of motion when each floor is displaced from its equilibrium position. The overall stiffness of the building depends on the stiffnesses of the structural members.

The first task is to derive the equation of motion for a single-story building. Then we examine both free and damped vibrations of the building. Next, the effects of a sinusoidal forcing function are considered in both the absence and presence of damping. With an understanding of the basic behavior of a single-story building, we proceed to multi-story buildings. Both forced and unforced vibrations are considered. Lastly, a three-story building is animated so that actual behavior of the building can be observed.

Unforced vibrations of a single-story building

Consider an idealized one-story building (Fig. 1) as an ordinary spring-mass-damper system (Fig. 2). Ignoring damping effects (omit the dashpots in Figs 1 and 2), standard arguments yield the following initial value problem (IVP) for the displacement of the center of mass of the roof:

$$mu'' + ku = 0, \quad u(0) = u_0, \quad u'(0) = u_1 \qquad (3)$$

The constants m and k represent the total mass of the roof and the overall stiffness of the walls, respectively. In the present setting, $k \propto EI/h^3$, where E is the elastic modulus of the supporting columns, I is the moment of inertia of the columns about the bending axis, and h is the length of the columns, (see [3], pp. 9–11 for more details). A natural prerequisite to understanding the behavior

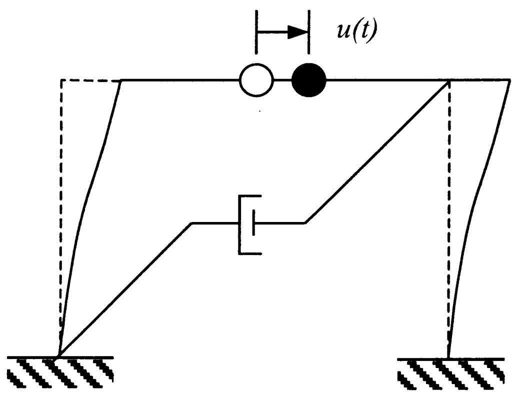

Fig. 1. Model of a single story building.

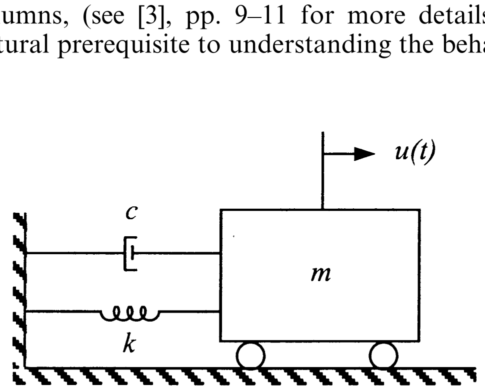

Fig. 2. Spring-mass-damper system.

of a building during an earthquake is to know how the building reacts to various initial conditions. Hence, we shall assume that the initial displacement u_0 and initial velocity u_1 are non-zero.

The IVP in equation (3) depends on two physical constants, m and k. More constants will be introduced later when damping and forcing are considered. Initially, it is convenient to ask the students to analyze the effects of the parameters on the system. This is obviously an open-ended question, especially as the model gets more realistic. However, it provides an opportunity to conduct parameter analysis which can be done easily using a CAS. More importantly, it motivates the utility of non-dimensionalizing the problem. Explicit dependence on m and k as well as dependence on u_0 is removed by simple changes of variables. Since $\sqrt{m/k}$ has units of time, let:

$$t = \sqrt{\frac{m}{k}}\tau, \quad u = u_0 y$$

Then the IVP in (3) reduces to the *nondimensional* IVP,

$$\ddot{y} + y = 0, \quad y(0) = 1, \quad \dot{y}(0) = v = \frac{u_1}{u_0}\sqrt{\frac{m}{k}} \tag{4}$$

where the dot indicates differentiation with respect to the dimensionless variable τ. The solution to (4) exhibits simple harmonic motion. Although this result is standard, it provides an opportunity to discuss conservation of energy and motivate phase plots. Specifically, multiplying the ODE in (4) by the dimensionless velocity $\dot{y}$ gives:

$$\frac{d}{d\tau}E(\tau) = 0, \text{ where } E(\tau) = \tfrac{1}{2}(\dot{y}^2 + y^2)$$

Requiring the students to interpret this result analytically (derivative of energy is zero), and graphically via a phase plot (circle centered at the origin), enables them to conclude that energy is conserved in the system. Note that for this particular system, the energy is proportional to the square of the distance from the origin to the trajectory. It is easy to see the exchange of kinetic and potential energy in the phase plane. When $\dot{y} = 0$ all of the energy is stored elastically by the spring and when y is zero, all of the energy is kinetic.

Following the traditional sequence of instruction, the students were asked to account for viscous damping which leads to the well-known IVP,

$$\ddot{y} + c\dot{y} + y = 0, \quad y(0) = 1, \quad \dot{y}(0) = v$$

where $c - \gamma/\sqrt{mk}$. We also asked them to *discuss* underdamped $(0 < c < 2)$, critically damped $(c = 2)$, and overdamped motion $(c > 2)$. All of these can easily be examined graphically both by plotting y versus τ and by making phase portraits.

Because most structures are underdamped, engineers often install friction devices in buildings [3] to minimize oscillations. These devices provide a damping mechanism similar to the friction caused by sliding a block along a flat dry surface. Imagine Fig. 2 without the wheels and dashpot. This is called Coulomb damping. The frictional force, $F_0 > 0$, opposes the direction of motion, but is *independent* of the magnitude of the velocity.

This was the only information given to the students who were asked to derive the governing IVP and find its solution. The IVP consists of a linear differential equation with a piecewise constant non-homogeneous term given in dimensional form by:

$$mu'' + ku = \begin{cases} -F_0 & \text{if } \dot{y} > 0 \\ F_0 & \text{if } \dot{y} < 0, \end{cases}$$
$$u(0) = u_0, \quad u'(0) = u_1 \tag{5}$$

with appropriate initial conditions. Alternatively, equation (5) can be expressed as an equivalent *nonlinear* IVP given in dimensionless form by:

$$\ddot{y} + y = -f_0\frac{|\dot{y}|}{\dot{y}}, \quad y(0) = 1, \quad \dot{y}(0) = v, \tag{6}$$

where $f_0 = (F_0/k|u_0|)$. While this ODE appears to be trivial, its solution is deceptively difficult to compute. First, it is not always valid. Whenever the velocity is zero, if the spring force is not greater than the frictional force, equation (6) does not apply and motion does not continue. That is, if $\dot{y} = 0$ and $|y| \le f_0$ (equivalently, $u' = 0$ and $|ku| \le F_0$), the building stops moving. Hence, motion is likely to stop when $y \ne 0$. Physically, this means that the building does not return to its original position! Moreover, this happens in finite time.

Note that multiplying (6) by $\dot{y}$ gives

$$\frac{d}{d\tau}E(\tau) = -f_0|\dot{y}| \le 0.$$

So friction causes energy to decay, but only if $\dot{y} \ne 0$. Thus, the energy will seldom decay to zero. Therefore, elastic energy remains stored in the structural members of the building after the vibrations stop.

When solving equation (6), one must carefully consider the initial conditions to determine if the velocity is initially positive or negative. Assuming $\dot{y}(0) = v > 0$, the non-homogenous term is negative so that the friction opposes the motion. One solution approach is to solve a sequence of initial value problems. Let $0 < r_0 < r_1 < \cdots < r_i < \cdots < r_n$ be the positive roots of the velocity function (note n must be finite). Also, let $y_0(\tau)$ be the solution on $[0, r_0]$, and let $y_i(\tau)$ be the solution to the IVP over the interval $[r_{i-1}, r_i]$ for $i \ge 1$. Since the non-homogeneous term is $-f_0$ for $\tau \in [0, r_0]$, we easily obtain:

$$y_0(\tau) = \sqrt{(1 + f_0)^2 + v^2}\cos(\tau - r_0) - f_0$$

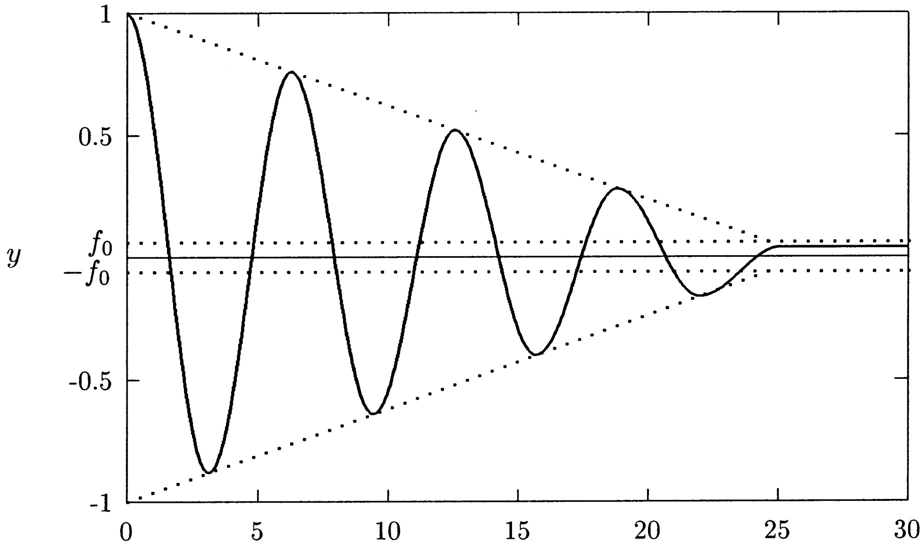

Fig. 3. Solution for a Coulomb damped spring-mass system (f_0 = .06).

$$\text{where } r_0 = \arctan\left(\frac{v}{1+f_0}\right) \tag{7}$$

Note that the final conditions for y_0 are the initial conditions for y_1, i.e.

$$y_1(r_0) = y_0(r_0), \quad \dot{y}_1(r_0) = \dot{y}_0(r_0) = 0$$

Since y_1 has the same period as y_0, $r_1 = r_0 + \pi$. Solving the IVP for y_1 and for successive y_i, we find:

$$y_i(\tau) = \left(\sqrt{(1+f_0)^2 + v^2} - 2if_0\right) \times \cos(\tau - r_0) + (-1)^{i+1} f_0, \quad i \geq 0 \tag{8}$$

The solution for $v < 0$ can be found in a similar fashion. This was the method typically employed by the students. Many students solved for the roots of $\dot{y}_i$ several times (usually numerically) before realizing that the roots were periodic. Of course, some simply never made the observation.

The solution persists until $|y_i(r_0 + i\pi)| \leq f_0$. Note from (8) that if this stopping criterion is not observed, then the solutions will eventually grow in amplitude! This makes students seriously consider the physics involved. A student who does not initially realize the stopping criterion is forced to do so because the solution is not physically possible when the amplitude increases.

A CAS is particularly convenient for visualizing the solution in different ways. Figure 3 shows y

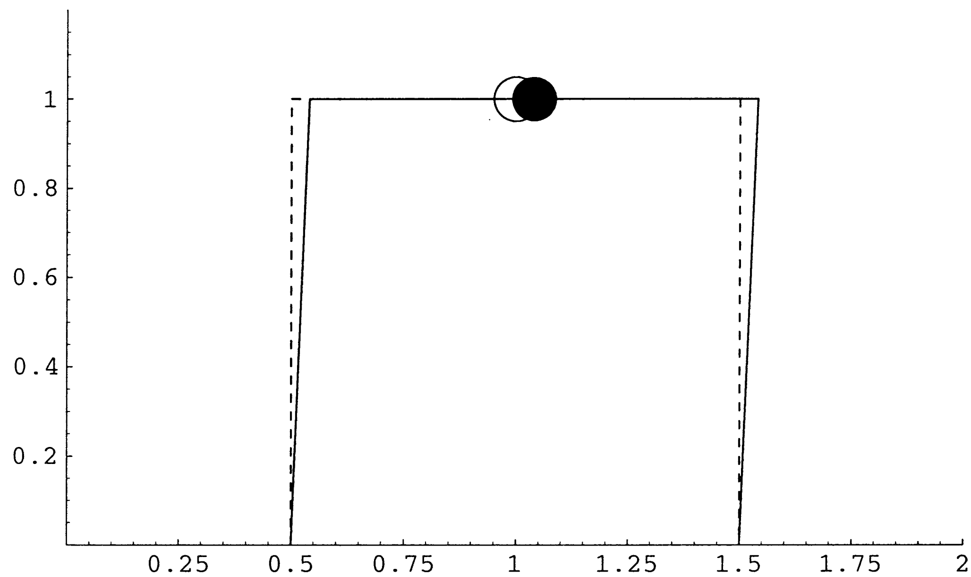

Fig. 4. Undisturbed (dashed line/empty circle) model of a single story building and the nontrivial equilibrium configuration for a Coulomb damped building for the solution given in Figure 3.

versus τ. Note that the final displacement is non-zero and satisfies the restriction $|y| < f_0$. In contrast to the exponential decay of viscous damping, the amplitude decays *linearly* as exemplified by the linear displacement envelope shown in Fig. 3. This is readily verified by computing the difference in amplitude over a full cycle of motion, i.e.

$$\begin{aligned} &|y(r_0 + i\pi) - y(r_0 + (i+2)\pi)| \\ &= |y_i(r_i) - y_{i+2}(r_{i+2})| = 4f_0 \end{aligned}$$

Furthermore, a CAS can be used to graph the final configuration of the building, as shown in Fig. 4, and create phase portraits and the corresponding vector field shown in Fig. 5. Here, it is obvious that energy remains stored in the spring since the trajectories do not go to the origin. Once a trajectory enters the darkened interval around the origin, $[-f_0, f_0]$, it never leaves.

We point out that although computer algebra systems have many capabilities, they are not flawless. Using the default options in NDSolve, *Mathematica* was often unable to compute solutions of equation (6) until the stopping criterion was reached. This reminds students of the limitations of technology.

The visualization techniques discussed above clarify the differences between viscous and Coulomb damping. Students often struggle with viscous damping since vibrations theoretically continue indefinitely, albeit with exponentially small amplitudes. However, they enjoyed Coulomb damping because the vibrations actually stop, which is consistent with their physical intuitions.

Forced vibrations

In the previous section, we modeled the unforced vibrations of a building for non-zero initial conditions. In this section, we consider the motion of the building during the earthquake (assuming harmonic forcing). Therefore, the building begins at rest in its upright position and the movement of the ground causes the building to move. If $u_G(t)$ is the horizontal displacement of the ground, then the total lateral deflection is $u_T(t) = u(t) + u_G(t)$ as indicated in Fig. 6.

Students were asked to derive the governing differential equation assuming a ground acceleration of $u''_G = -A\sin(\omega\sqrt{k/mt})\,\mathrm{m/sec^2}$. Without damping, this gives $mu''_T + ku = 0$, which implies:

$$mu'' + ku = -u''_G = A\sin\left(\omega\sqrt{\frac{k}{m}}t\right) \tag{9}$$

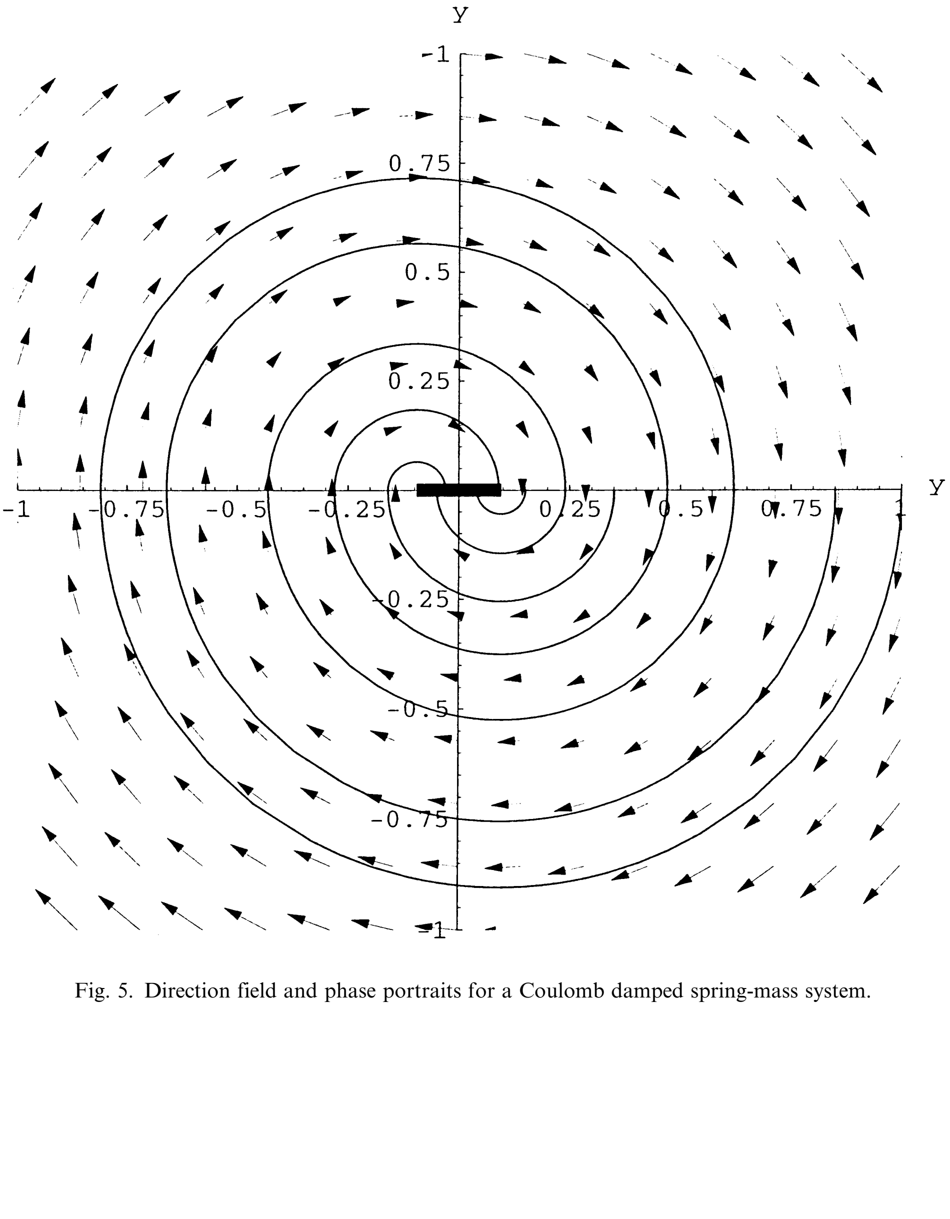

Fig. 5. Direction field and phase portraits for a Coulomb damped spring-mass system.

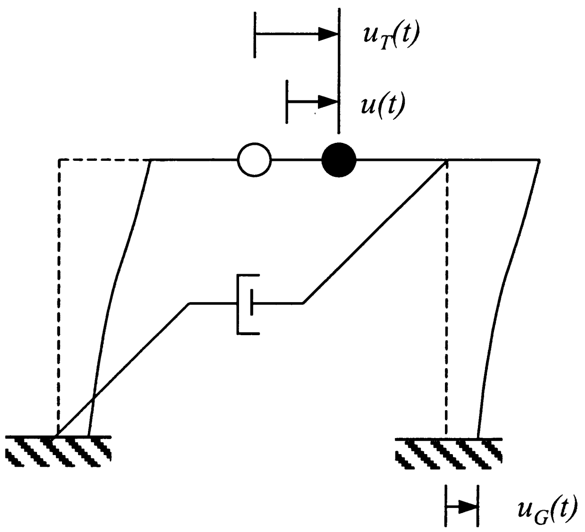

Fig. 6. Building model with ground motion.

By letting $t = \sqrt{m/k}\tau$ and $u = (A/k)y$, we can rewrite this equation in nondimensional form as:

$$\ddot{y} + y = \sin(\omega\tau), \quad y(0) = 0, \quad \dot{y}(0) = 0. \tag{10}$$

Note that u is scaled differently than in the previous section since the initial displacement $u_0 = 0$. The IVP (10) now depends solely on the parameter of interest ω.

Students were required to solve the ODE in (10), plot the solutions for various values of ω, and discuss the behavior of the solutions. This was a rather open-ended question and hence responses varied widely. However, it was an opportunity for students to explore the concepts of beats, resonance, and long-term behavior on their own while wrestling with the open-ended structure of the problem.

Next, we asked the students to consider viscous and Coulomb damping. The treatment for viscous damping is well known. However, for Coulomb damping, the building does not move unless the resultant of the spring force and external force are larger than the friction force, i.e., $|ku - A\sin(\omega\sqrt{k/m}t)| > F_0$. Thus, it is possible for motion to stop temporarily and restart when the resultant force becomes large enough. This phenomenon is called **stick-slip**, see [8] for more details. It is another example of a problem for which standard numerical solvers can fail, but physical intuition can be used to overcome their limitations.

Finally, students were asked:

1. If the building is designed to withstand deflections of less than 0.75 m, how long could an undamped building endure forced vibrations at the resonant frequency?
2. Determine the smallest value of the viscous damping coefficient, c, that prevents catastrophic failure if the building is forced at the resonant frequency $\omega = 1$.
3. Determine the magnitude of the force required for a Coulomb-damped building to begin oscillating. Does the frequency of the applied force matter?

Multi-story buildings

In this section, we model the structural dynamics of a building consisting of three floors, although the analysis easily generalizes to n stories. As in the single-story case, the mass of each floor is modeled as though it were concentrated at its center of mass, so the problem essentially reduces to the interaction of three masses as shown in Fig. 7. For convenience, assume that the floors have equal mass, all structural members have equal stiffness, and the physical constants and ground acceleration are the same as given above.

Students were expected to derive and analyze a non-dimensionalized system of equations for both free and forced vibrations. The model is derived from standard arguments, but differs slightly from presentations in traditional ODE textbooks since one end of the spring/mass system is free. If we ignore damping (no dashpots in Fig. 7) and let u_i be the displacement of the i^{th} floor, then the governing system of ODE's in matrix form is given by:

$$\ddot{\boldsymbol{u}} + \boldsymbol{A}\boldsymbol{u} = \boldsymbol{f}(t),$$

$$\text{where } \boldsymbol{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}, \quad \boldsymbol{A} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} \tag{11}$$

and $\boldsymbol{f}(t)$ is an applied external force.

With $\boldsymbol{f}(t) = \boldsymbol{0}$, equation (11) strongly resembles the scalar equation (4). Thus, it is reasonable to seek solutions of the form $\boldsymbol{u} =$

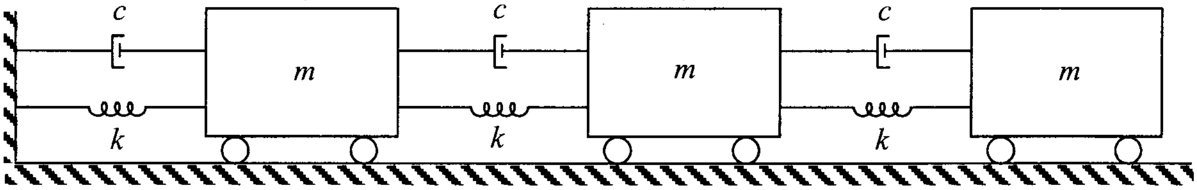

Fig. 7. Spring-mass-damper system with 3 masses.

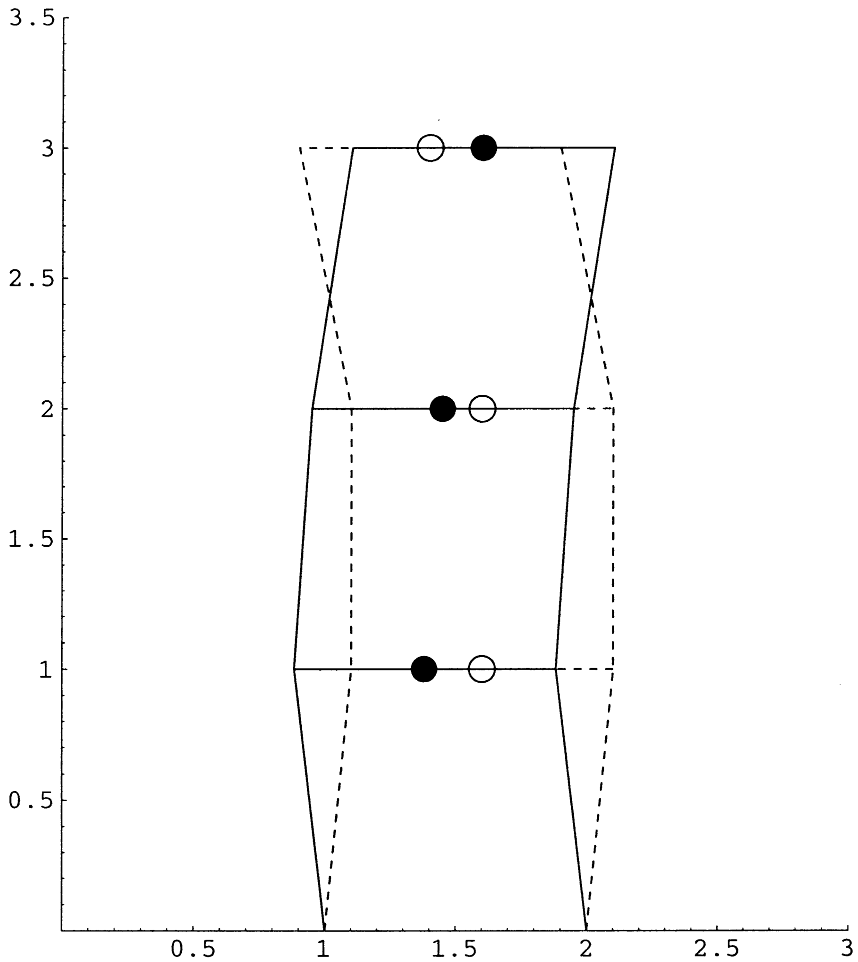

Fig. 8. Two snapshots from the animation of a vibrating three-story building.

$\xi(c_1 \sin r\tau + c_2 \cos r\tau)$. This yields the eigenvalue problem

$$A\xi = r^2\xi \tag{12}$$

Because the top of the building is free, the eigenvalues are not simple to compute. It is convenient to find the eigenvalues numerically with a CAS. The initial conditions:

$$u_1(0) = u_2(0) = 0.1; \quad u_3(0) = -0.1;$$
$$\dot{u}_1(0) = \dot{u}_2(0) = \dot{u}_3(0) = 0 \tag{13}$$

were specified and students were asked to graph their solutions.

Again, the utility of a CAS is obvious. Furthermore, actual animations of the model building are easy to produce. The following *Mathematica* commands numerically solve equations (11) and (13) and produce an animation of a three-story building. Two snapshots of this animation are shown in Fig. 8.

```
sol=NDSolve[{u1''[tau]+2 u1[tau]-u2[tau]=
  =0,u2''[tau]-u1[tau]+2 u2[tau]-u3[tau]=
  =0, u3''[tau]-u2[tau]+u3[tau]==0,u1[0]=
  =.1, u1'[0]==0, u2[0]==.1, u2'[0]==0,
  u3[0]==-.1, u3'[0]==0}, {u1, u2, u3},
  {tau, 0, 30}]
x1=1+First[Evaluate[u1[tau]/.sol]];
x2=1+First[Evaluate[u2[tau]/.sol]];
x3=1+First[Evaluate[u3[tau]/.sol]];
tb={{1,0},{x1,1},{x2,2},{x3,3},{1+x3,3},
  {1+x2,2},{1 + x1,1},{2,0},{1 + x1,1},
  {x1,1}, {x2,2}, {1+ x2,2}};
Do[ListPlot[tb,PlotJoined->True,
  PlotRange->{{0,3},{0,3.5}},
  AspectRatio->Automatic,
  Epilog->{Disk[{x1+.5,1},.08],
  Disk[{x2+.5,2},.08],
  Disk[{x3+.5,3},.08]}], {tau,0,30,.2}]
```

A major difference between the multiple-story and single-story buildings is that simple harmonic motion does not occur in general for a system. Only if the initial conditions happen to coincide with an eigenvector of A will simple harmonic motion result. These eigenvectors mathematically represent the different possible modes of vibration, and any free vibration of the multiple-story building is a linear combination of these modes. It was quite enlightening for students to visualize the vibration modes via an animation. This traditionally difficult concept became quite clear almost immediately.

The next topic of the project involved sinusoidal forcing for the multi-story case. If the bottom floor is accelerated by the ground, then $\boldsymbol{f}(t) = (1, 0, 0)^T \sin \omega t$. Students generalized their knowledge of the method of undetermined coefficients or variation of parameters to systems by referring to a standard text such as [2]. Alternatively, solutions could simply be found numerically. Although $\omega = 1$ is the resonant frequency for the single-story building discussed earlier, it is not a resonant frequency for the three-story building. Students were asked to discuss why this is the case from both mathematical and physical perspectives. Finally, the resonant frequencies were computed and solutions were obtained for a building which begins at rest in its upright position. Again, animations of the solutions were quite valuable for developing physical intuition. Students were able to "see" the eigenvectors in the excited modes of vibration.

SUMMARY AND EXTENSIONS

We found that the project was extremely effective at developing several mathematical skills among the students. It targeted several mathematical concepts used to study ODE's and gave students the opportunity to explore concepts on their own, address open-ended questions, realize physical limitations of mathematical models, encounter some of the limitations of numerical solvers, enforce and develop their linear algebra skills, enhance their skills with CAS's, and further develop their understanding of several concepts in ODE theory and interdisciplinary applications. The animations were extremely effective at developing their understanding of vibration phenomena and only required a small investment of time. Perhaps the most gratifying result was that many students commented that this project really forced them to think!

We reiterate that the basis of this project was designed for students taking a Calculus with Differential Equations course, which is required for all cadets at USMA. As such, the amount of ODE theory covered was significantly less than what one would typically find in a course devoted solely to differential equations. In particular, at this level, the primary goals of the course involved first-order ODE's and second-order differential equations which could be solved using the method of undetermined coefficients and equivalent systems of ODE's. Additionally, there was a brief introduction to numerical solutions to differential equations. Needless to say, we believe that the students were much more prepared to be successful in their engineering and applied mathematics courses as a result of completing this project.

There are a tremendous number of possible extensions to this project which can be used to develop mathematicians and engineers. One possibility is to consider different forcing functions such as delta functions, piecewise continuous functions, etc., which motivate Laplace transforms for scalar equations and for systems. Also, since sinusoidal forcing does not approximate the motion of an earthquake very accurately, actual ground motion data can be obtained and solutions derived numerically. Of course, this can naturally lead to many discussions involving Fourier analysis. Several internet sites provide sources for obtaining actual data [9, 10], and links therein for additional information.

Another possibility is to incorporate damping in the multi-story model. Viscous damping leads to a system of the form:

$$\boldsymbol{M}\boldsymbol{u}'' + \boldsymbol{C}\boldsymbol{u}' + \boldsymbol{K}\boldsymbol{u} = \boldsymbol{0} \tag{14}$$

where $\boldsymbol{M}$ is a mass matrix, $\boldsymbol{K}$ is a stiffness matrix, and $\boldsymbol{C}$ is a damping matrix. Noting the similarity of this system to equation (1) and seeking solutions of the form $\boldsymbol{u} = \xi e^{rt}$, leads to a generalized eigenvalue problem:

$$(\boldsymbol{M}r^2 + \boldsymbol{C}r + \boldsymbol{K})\xi = 0 \tag{15}$$

Alternatively, the system in equation (14) can be cast as a first-order system. See [6] for a discussion. With Coulomb damping, floors may stop and start several times before the entire building comes to rest.

The interested reader should consult [3] and [4] for other interesting topics such as base isolation, inelastic phenomena, and vibrations of structures with distributed mass, e.g. chimneys, cooling towers, etc.

REFERENCES

1. P. Blanchard, R. L. Devaney and G. R. Hall, *Differential Equations*, Brooks/Cole Publishing Company, Pacific Grove, California (1998) pp. 146–149.
2. W. E. Boyce and R. C. DiPrima, *Elementary Differential Equations and Boundary Value Problems*, John Wiley & Sons, Inc., New York (1997) pp. 179–201.
3. A. K. Chopra, *Dynamics of Structures, Theory and Applications to Earthquake Engineering*, Prentice Hall, Inc., Upper Saddle, New Jersey (1995).
4. R. W. Clough and J. Penzien, *Dynamics of Structures*, 2nd ed., McGraw-Hall, Inc., New York (1993).
5. P. W. Davis, *Differential Equations, Modeling with Matlab*, Prentice Hall, Inc., Upper Saddle River, New Jersey (1999) pp. 276–292.
6. C. H. Edwards, Jr. and D. E. Penney, *Differential Equations, Computing and Modeling*, Prentice-Hall, Inc., Englewood Cliffs, New Jersey (1996) pp. 302–304.

7. A. Gray, M. Mezzino and M. A. Pinsky, *Introduction to Ordinary Differential Equations with Mathematica*, Springer-Verlag, New York (1997) pp. 325–376.
8. S. G. Kelly, *Fundamentals of Mechanical Vibrations*, McGraw-Hill, Inc., New York (1993) pp. 131–134.
9. The Multidisciplinary Center for Earthquake Engineering Research, SUNY, Buffalo, http://mceer.buffalo.edu/.
10. The National Information Service for Earthquake Engineering, University of California, Berkeley, http://www.eerc.berkely.edu/.
11. R. E. O'Malley, Jr, *Thinking About Ordinary Differential Equations*, Cambridge University Press, Cambridge, United Kingdom (1997) pp. 61–65.
12. D. G. Zill, *Differential Equations with Computer Lab Experiments*, Brooks/Cole Publishing Company, Pacific Grove, California (1998) pp. 172–190.

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