

# Modelling and Computation Techniques for Fluid Mechanics Experiments\*

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*A method for describing and analysing an experiment involving the efflux time from a vessel is given. It follows a process modelling approach together with numerical solution and parameter identification using a spreadsheet. The method produces a flexible model that can be applied to a variety of fluid mechanics problems and teaches a generic set of tools useful to practising engineers.*

## INTRODUCTION

THE AVAILABILITY of a wide range of computational tools has had a significant impact on the manner in which engineering problems can be described, solved and hence taught. Van Dongen and Roche [1] recently presented an undergraduate experiment in fluid mechanics in which the efflux time was measured and compared with theory.

In this paper an alternative method is presented to describe and quantify such an experiment. This alternative makes greater use of the computational tools available and teaches a more generic set of skills that are useful to a practising engineer. Not only does it teach elements of fluid mechanics, but it can be used for concepts of modelling, simulation and process dynamics.

The experiment and associated lectures use the general teaching philosophy that, where possible, methods of analysis should be applicable to a range of problems, be based on physical phenomena, and support other parts of the students' learning. Specifically in this case the skills being developed are:

- the application of Bernoulli's equation and friction equations to flow systems;
- a range of spreadsheet skills including writing formulae, VLookup, Solver and documentation;
- dynamic modelling of a system;
- numerical methods for the solution of simple dynamic systems.

### Apparatus and procedure

An apparatus is set up consisting of a tank with an attached exit pipe as shown in Fig. 1. This is based on both the apparatus presented by van Dongen and Roche, and a similar apparatus in the current author's department.

Attached to the tank are a known length of tubing and fittings. An attached ruler can be

used to measure the height of liquid and a stopwatch can be used to measure time.

The tank is filled with liquid with the exit pipe closed. The exit is opened and a series of height and time readings are recorded. It can be useful to use the lap time on the stopwatch to measure the times at which a series of predetermined heights are reached.

The known parameters of the system are measured and recorded.

### Process model

The process can be described by standard and fundamental relationships. The development of the model can be an exercise in itself but, if the primary object is one of teaching fluid mechanics, the model can be given to the students. However as an example of a model it should include an objective, scope, diagram, assumptions, equations, list of variables and its range of validity.

The objective of the model is to be able to determine the height of liquid in the tank as a function of time. Assuming no significant changes in temperature, the density will be constant and uniform.

### Equations

With constant and uniform density, the mass balance becomes a volume balance:

$$\frac{dV}{dt} = F_1 - F_2 \quad (1)$$

The inlet flow,  $F_1$ , is usually zero but may be a known function of time. The outlet flow is given by:

$$F_2 = u_2 A_2 \quad (2)$$

From the volume in the tank we can get the height of liquid in it:

$$h_1 = \frac{V}{A_1} \quad (3)$$

\* Accepted 5 August 2000.

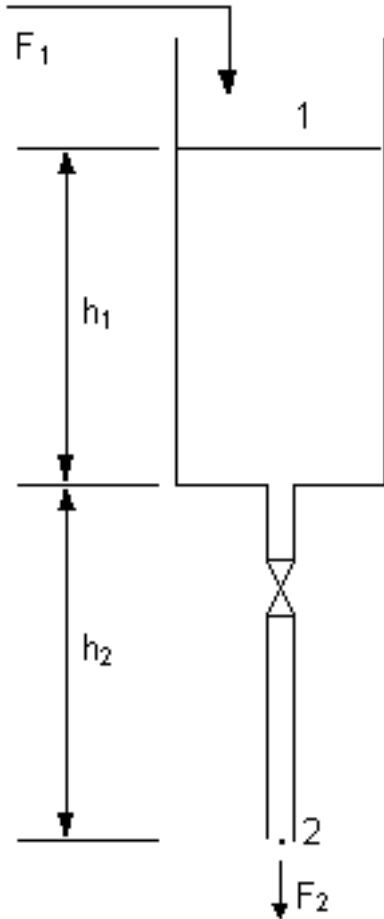


Fig. 1. Experimental apparatus.

Bernoulli's equation including the friction pressure drop applies. It can be expressed, noting points 1 and 2 on Fig. 1, as:

$$P_2 - P_1 + \rho \left( \frac{u_2^2 - u_1^2}{2} \right) + \rho g(z_2 - z_1) = -\Delta P_{friction} \quad (4)$$

where the friction pressure drop is given by:

$$\Delta P_{friction} = \left( \frac{4fL}{D_2} + \sum_{fittings} k \right) \frac{\rho u_2^2}{2} \quad (5)$$

The velocity of liquid within the tank,  $u_1$ , can be considered negligible. (This assumption leads to a simpler solution method for the equations.) Combining these two equations we get the expression:

$$\left( 1 + \frac{4fL}{D_2} + \sum_{fittings} k \right) \frac{\rho u_2^2}{2} = \rho g(z_1 - z_2) + P_1 - P_2 \quad (6)$$

from which  $u_2$  can be determined explicitly:

$$u_2 = \sqrt{\frac{2(\rho g(z_1 - z_2) + P_1 - P_2)}{\rho \left( 1 + 4fL/D_2 + \sum_{fittings} k \right)}} \quad (7)$$

Noting the signs carefully, the height  $z_1$  is defined:

$$z_1 = z_2 + (h_1 + h_2) \quad (8)$$

The friction factor can be calculated within the accuracy of the original data using equation (9) of Haaland [2] which is claimed to give values within 1% of the Colebrook and White formula [3] for  $Re > 5000$ , and is still within 3% of the Moody chart at a Reynolds' number of 3000.

$$f = \left( 1 / -3.6 \log_{10} \left[ \left( \frac{\epsilon}{3.7D_2} \right)^{1.11} + \frac{6.9}{Re} \right] \right)^2 \quad (9)$$

The equation of Churchill, as given by Perry and Green [4], gives similar results. The Reynolds' number is defined as usual:

$$Re = \frac{D_2 u_2 \rho}{\mu} \quad (10)$$

Given the diameter of the exit, the area is defined:

$$A_2 = \frac{\pi D_2^2}{4} \quad (11)$$

We now have a set of eight equations (1) to (3) and (7) to (11) that describe the system.

#### Variables

The variables must also be defined for our description of the process to be complete. These are grouped into different types of variables:

#### Independent variable

Re	$t$ time	s
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#### Dependent variables

$A_2$	Cross-sectional area of the exit pipe	$m^2$
$f$	Fanning friction factor	
$F_2$	Flow rate of liquid leaving the exit	$m^3 s^{-1}$
$h_1$	Height of liquid in the tank	m
Re	Reynolds' number	
$u_2$	Velocity of liquid leaving the tank	$m s^{-1}$
$V$	Volume of liquid in the tank	$m^3$
$z_1$	Height of liquid relative to $z_2$	m

#### Inputs

$F_1$	Flow rate of liquid entering the tank	$m^3 s^{-1}$
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#### Parameters

$A_1$	Cross-sectional area of the tank	$m^2$
$D_2$	Diameter of exit pipe	m
$h_2$	Vertical distance from the tank base to the pipe exit	m
$k$	Resistance coefficients for the fittings	m
$L$	Length of the exit pipe	m
$P_1$	Pressure at the top of the tank	Pa (probably zero)

$P_2$	Pressure at the exit	Pa (probably zero)
$z_2$	Height of exit	m (define as zero)
$\epsilon$	Pipe roughness	m
$\mu$	Liquid viscosity	Pas
$\rho$	Liquid density	$\text{kg m}^{-3}$

an approximate numerical solution can be obtained from:

$$V(t_i) = V(t_{i-1}) + \Delta t \left. \frac{dV}{dt} \right|_{t_{i-1}} \quad (12)$$

where

$$t_i = t_{i-1} + \Delta t, \text{ for } i = 0, n \quad (13)$$

An initial condition is required for the volume, although experimentally it is easiest to measure the liquid height and calculate the volume from equation (3).

The model is valid while the Reynolds' number is greater than 3000, and while the liquid height is between and top of the tank and the exit point in the tank.

We note that there are eight equations and eight dependent variables, which is a necessary but not sufficient condition that the system has been correctly modelled.

For this particular system of equations we can write a spreadsheet as shown in Fig. 2 to solve the equations. The model equations are used as indicated on the sheet, but the initial values of the variables are set to known initial conditions as noted. The equations can be copied down from the second row for as many rows as required to obtain a solution. Where a parameter is unknown, a guess can be made so that a solution can be obtained.

*Numerical solution of the model*

The solution of the model can be an exercise in itself, but again, if the objective of the exercise is one of fluid mechanics only, the solution spreadsheet can be developed for the students to use. In this manner they get exposure to a solution technique without having to know all the detail.

This model is easily solved using Euler's method on the spreadsheet as described, e.g., by Orvis [5]. Given a differential equation, e.g.

$$\frac{dV}{dt} = F_1 - F_2, \text{ given } V(t_0) \quad (1)$$

*Parameter identification*

In this particular model all the parameters are known except the resistance coefficient,  $k$ , for the constriction in the exit pipe. This can be determined by trial and error to find the value that gives the best visual fit of experimental and model results when plotted on the same graph.

A more rigorous method is to minimise the sum of the squared error between the experimental measurements and the numerical solution. This can be done using a number of features of Excel starting with the VLookup function as shown in Fig. 3. If small time steps are used for

	A	B	C	D	E	F	G	H	I
1	<b>Tank Efflux Model</b>		Ken Morison		31-Jan-00				
2									
3	<b>Parameters</b>				<b>Fittings</b>				
4	Tank diameter	D1	0.0845	m	Constriction		k_constr	3.966	
5	Exit diameter	D2	0.0056	m	Total		k	3.966	
6	Pipe exit height	h2	0.242	m	<b>Time invariant variables</b>				
7	pipe length	L	0.205	m	Tank area		A1	0.00561	m <sup>2</sup>
8	Tank pressure	P1	0	Pa gauge	Exit Pipe Area		A2	2.46E-05	m <sup>2</sup>
9	Exit pressure	P2	0	Pa gauge	<b>Euler's method</b>				
10	Pipe roughness	e	0	m	Time step		dt	1	s
11	Density	rho	998	kg/m <sup>3</sup>					
12	Viscosity	mu	0.001	Pa.s					
13	Gravity	g	9.81	m/s <sup>2</sup>					
14									
15	t	V	h1	z1	u2	F2	Re	f	dV/dt
16	s	m <sup>3</sup>	m	m	m/s	m <sup>3</sup> /s			m <sup>3</sup> /s
17	equn (13)	equn (12)	equn (3)	equn (8)	equn (7)	equn (2)	equn (10)	equn (9)	equn (1)
18	0	0.001037	0.185	0.427	1.159	0.0000285	6476	0.0087	-0.0000285
19	1	0.001009	0.180	0.422	1.151	0.0000284	6435	0.0087	-0.0000284
20	2	0.000981	0.175	0.417	1.144	0.0000282	6395	0.0087	-0.0000282
21	3	Notes: (1) t0 is set to zero							0.00280
22	4	(2) V(t0) is calculated from h1 and equn (3)							0.00278
23	5	(3) h1(t0) is set to the measured height at t = 0							0.00277
24	6	(3) To avoid iteration f is calculated using Re from the previous step							0.00275
25	7	(4) The initial value of f is guessed and can be updated after typical values are found							0.00273
26	8	0.000814	0.145	0.387	1.101	0.0000271	6156	0.0088	-0.0000271

Fig. 2. Numerical solution of the model using Euler's method on a spreadsheet. (Equation numbers in row 17 correspond to those in this paper.)

	J	K	L	M	N	O	P	Q
1	<b>Experimental Results</b>							
2								
3		<b>Experimental</b>		<b>Model</b>	<b>Difference<sup>2</sup></b>			
4		<b>Time</b>	<b>h1</b>	<b>h1</b>	<b>(h1-h1<sub>model</sub>)<sup>2</sup></b>			
5		s	m	m				
6								
7		0	0.1850	0.1850	0.00E+00			=VLOOKUP(K8,solution,3)+ (K8-VLOOKUP(K8,solution,1)) *VLOOKUP(K8,solution,9)/\$H\$9
8		3.99	0.1650	0.1649	1.34E-08			
9		8.02	0.1450	0.1451	5.46E-09			"solution" is the name of the range A18:190" in figure 2.
10		12.08	0.1250	0.1256	3.97E-07			Use Solver to set the target cell N17 (the sum of squared differences) to a minimum by changing the value of k for the constriction (H4).
11		16.49	0.1050	0.1051	9.17E-09			
12		20.92	0.0850	0.0851	5.36E-09			
13		25.58	0.0650	0.0647	1.05E-07			
14		30.24	0.0450	0.0450	2.39E-09			
15		35.14	0.0250	0.0249	3.94E-09			
16								
17		Sum of Squared differences			5.42E-07			

Fig. 3. Experimental results are compared to the model results.

the numerical solution, VLookup can be used directly to find the predicted liquid height for a given experimental time. However the value obtained can be improved by using linear interpolation. The interpolated height,  $h_{int}$ , can be found at a required time,  $t_{int}$ , from a neighbouring set of values of time, height and derivative ( $t_o$ ,  $h_o$ , and  $dh/dt$ ) in the numerical solution.

$$h_{int} = h_o + (t_{int} - t_o) \frac{dh}{dt} \quad (14)$$

This can be implemented in Excel using a simple but perhaps messy looking formula that is explained here in parts. If, as in Fig. 3, the experimental time is in cell K8, the area  $A_1$  is in cell H9 (Fig. 2) and the numerical solution have been named as the range 'solution' (A18:190 in Fig. 2), the set of values of time, height and derivative can be obtained respectively using:

$$h_o = \text{VLOOKUP}(K8, \text{solution}, 3)$$

$$t_o = \text{VLOOKUP}(K8, \text{solution}, 1)$$

$$dh/dt = \text{VLOOKUP}(K8, \text{solution}, 9) / \$H\$9$$

Thus the entire cell formula (in M8) cell for the interpolated height is:

$$= \text{VLOOKUP}(K8, \text{solution}, 3) + \\ (K8 - \text{VLOOKUP}(K8, \text{solution}, 1)) * \\ \text{VLOOKUP}(K8, \text{solution}, 9) / \$H\$9$$

Once the corresponding liquid heights from the numerical solution have been found for the experimental times, the sum of the squared difference between experimental and model values is calculated. Solver is then used in the manner described by John [6] to minimise the sum of squared differences by changing the cell for the unknown

parameter(s). In this case  $k$  for the constriction is the variable that is changed by Solver to give the best fit.

#### Validation

The model equations and the parameters can be verified by carrying out a similar but different experiment. If, for example, the exit pipe is shortened and the new values of  $L$  and  $h_2$  are entered into the sheet, the model should accurately predict the change in height with time for the new pipe. If it does not, the parameters and assumptions need to be checked.

#### Extensions of the experimental, model and simulation

There are many extensions that can be made to the experiment and to the model, to aid understanding of fluid mechanics or of modelling and simulation.

Some fluid mechanics enhancements can be done experimentally. It has been found effective to get a set of predictions from students before attempting the experiments. It encourages students to generate and propose ideas even if they are not correct. Enhancements include:

- use a variety of fittings and pipe lengths and diameters;
- use a rough pipe;
- consider the effects of a side exit rather than a bottom exit;
- check the difference between square and rounded contractions at the exit;
- use a closed vessel so flow occurs until sufficient vacuum is produced;
- use a different fluid [1].

Modelling and simulation enhancements:

- change the vessel shape, e.g. to conical;
- model the flow from a closed system: pressure terms and the gas law will need to be included;
- modify the model so that it is valid for lower Reynolds' numbers;
- check the effect of step size on the accuracy of Euler's method;
- write the solution method as a Visual Basic program that can be used from Excel;
- use another numerical integration technique, e.g., modified Euler or Runge Kutta [7].

Enhancements for teaching process dynamics:

- add an inlet flow as a pulse or step change;
- duplicate the model to get a model for two tanks in series or parallel.

In a similar exercise students are asked to predict whether a system with a long exit pipe will drain faster than one with a short exit pipe. The correct answer depends on the physical dimensions of the system but it does encourage some good thinking. The students work through a sequence of observation, prediction, experimentation, analysis and interpretation. This approach has worked well especially when predictions are not the same as the observations.

## DISCUSSION

The approach outlined above has numerous benefits. For teaching fluid mechanics the students see a range of equations that will be useful in many different situations. By keeping algebraic manipulation to a minimum the fundamental relationships are not lost. In contrast most of the equations presented by van Dongen and Roche [1] are applicable only to efflux time and not to other fluids problems.

The use of the equation (9) of Haaland [2] has a greater range of validity than that of Blasius, which is applicable for turbulent flow through smooth pipes only. While the Haaland equation is more complex, once written into a spreadsheet and checked, it is more useful. The historical popularity of the Blasius equation may have been because of the ease of calculating results on a slide rule.

Friction pressure drop is given here in terms of

resistance coefficients rather than equivalent length as used by van Dongen and Roche. The concept of equivalent length implicitly suggests that pressure drop through fittings is related to the friction in a pipe. However theory for pressure drop through expansions and contractions naturally leads to the 'velocity head' form used in equation (5). While the concept of equivalent length enabled fewer calculations in the past, the use of spreadsheets enables the more rigorous 'velocity head' method to be used with ease.

During the teaching of modelling, it is emphasised that a model is much more than a set of equations, and that it also includes the model objective, scope, assumptions, variables, initial conditions and range of validity.

When the students are asked to model a different but related system, they see the value in keeping algebraic manipulation to a minimum. They also see the value of thinking ahead to other possible uses of the model. As examples, the model allows examination of systems with an inlet flow also (equation (1)). If the system was under pressure or vacuum, equation (7) already contains the required terms and, if necessary, one or two other equations could be added to describe changes in the pressure with time or liquid level. The model also applies to tank filling, though care must be taken with the numerical sign of the variables.

The dynamic modelling approach clearly defines time as the independent variable. It is volume and height that depend on time, rather than time that depends on height as implied by equation (1) of van Dongen and Roche [1]. The tendency for students to try to obtain explicit expressions for time dependence seems to inhibit their understanding of dynamic systems. Modelling the system with the differential equation leads much more naturally on to other problems of dynamics, especially for process control studies.

The use of the generalised least-squares approach using Solver [6] with a spreadsheet eliminates the need to find an expression that will give a straight line. This approach can be applied to a wide range of problems without the need for manipulation or simplification of the fundamental equations.

The overall exercise uses skills in fluid mechanics, modelling, numerical solutions and spreadsheeting that can be expected from modern graduates in many fields of engineering.

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## APPENDIX

### *Spreadsheet formulae*

Typical formulae in the spreadsheet shown in Fig. 2 are given in the following table.

Table A1. Typical formulae in Fig. 2

Cell	Formula
A19	= A18+H\$13
B18	= C18*H\$9
B19	= B18+I18*(A19A18)
C19	= B19/H\$9
D18	= C18+C\$6
E18	= SQRT((\$C\$11*\$C\$13*D18+\$C\$8\$C\$9)/(1+4*H18*\$C\$7/\$C\$5+H\$6)*2/\$C\$11)
F18	= E18*H\$10
G18	= \$C\$5*E18*\$C\$11/\$C\$12
H19	= 1/(-3.6*LOG10((\$C\$10/\$C\$5/3.7)^1.11+6.9/G18))^2
I19	= -F18
H9	= PI()/4*C4^2
H10	= PI()/4*C5^2

Cells in columns A to I below these cells are copied down from these cells.

**Ken Morison** a senior lecturer in the Department of Chemical and Process Engineering. He teaches spreadsheeting, programming, modelling, simulation techniques and fluid mechanics to engineering students. Wherever possible he aims to use calculation techniques that take advantage of the computations and modelling skills that students are learning, and that they need as engineers.