

# Root Computations of Real-coefficient Polynomials using Spreadsheets\*

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*Two spreadsheet solutions of the roots of linear real-coefficient polynomials of any order are presented. The first procedure is a combination of the Routh-Hurwitz method and the bisection method. The second procedure is based on Bairstow's method. Both procedures are reliable in determining the real, complex-conjugate, distinct, and repeated roots of the polynomial. The user simply enters the coefficients of the polynomial and the initial guesses, and the  $n$  roots will be calculated and displayed on the worksheet within specified precision. Examples are given in both cases for illustration and for comparison.*

## AUTHOR QUESTIONNAIRE

1. The paper describes software tools suitable for students of the following courses:  
Numerical Analysis and control systems.
2. Level of students involved in the use of the materials:  
Junior and senior.
3. What aspects of your contribution are new?  
The procedure for computing the roots of a polynomial using the RH criterion.
4. How is the material presented to be incorporated in engineering teaching?  
Can be introduced as a new numerical procedure for polynomial roots computation.
5. Which texts or other documentation accompany the presented materials?  
The new procedure will soon be available on a web site.
6. Have the concepts presented been tested in the classroom? What conclusions have been drawn from the experience?  
Yes, and comparison with other numerical solutions are obtained and explained in the paper.
7. Other comments on benefits of your presented work for engineering education:  
The new procedure can be given as a part of the material covered in numerical analysis courses for solving for the roots of a polynomial. The same procedure can be outlined in a feedback control course when the stability of the system is under consideration.

## INTRODUCTION

BRACKETING METHODS and open methods are the standard methods for determining roots of polynomials. These methods are designed to determine the value of a single root on the basis of foreknowledge of its approximate location. Alternatives for calculating an approximate location of a root are very limited. One method to obtain an approximate value for a root is to plot the function and determine where it crosses the x-axis. The need for a long time and large space, and the lack of precision and reliability discounts are drawbacks of the graphical methods.

Other alternatives are the methods that use successive derivations to bracket the root. A real single or multiple root of a polynomial is always located between a maximum and a minimum of different signs or between an extremum and infinity. Therefore, in order to bracket the root, the abscissa of the extremum points should be located. This could be done by solving the derivative of the initial polynomial for the extremum. Being a root of the derivative, the extremum is bracketed within a maximum and a minimum which abscissa values are located by a further iteration. These methods have two major drawbacks: The inability to get the complex roots when they exist and their complexities [1, 2].

Routh-Hurwitz [3] in combination with shifting can incorporate the role of incremental search in a short time. The incremental search is done to determine the locations of all roots. The best way for getting the initial guess is by locating all roots after the first shift, in one half of the complex plane. Once the initial guess is found, a small increment in the shifting procedure is used. Once the shifting is equal to the real roots or to the real

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part of a complex conjugate pair of roots, the Routh table displays some special characteristics. These characteristics are closely watched and used to display the roots of the polynomial. Repeated application of this method, always result in closer estimates of the true value of the roots. Such a method is said to be convergent.

The Bairstow method divides the original polynomial of order  $n$  by a quadratic factor of the form:  $x^2 - rx - s$  allowing the determination of complex roots. If this is done, the result is a new polynomial of order  $(n - 2)$  with a remainder of the form  $R = b_1(x - r) + b_0$ . Thus, the method reduces to determining the values of  $r$  and  $s$  that make the quadratic factor an exact divisor. Bairstow's method uses a strategy similar to the Newton-Raphson approach to improve the original guess thus obtaining the roots of the quadratic factor or setting the remainder term to zero.

In this paper, Excel is used to determine the roots of a real-coefficient polynomial of order greater than or equal to 3. Numerical examples for a polynomial of order 8 are given for illustration. The advantage of the spreadsheet method is its generality and its use of a readily available software tool. It is also worth mentioning that applying a general spreadsheet program saves time and money compared to developing a new dedicated software package. It allows students to have first-hand experience on design problems by using the characteristics of the spreadsheet. Parameters can be changed quickly and new values will appear immediately. It is therefore possible to do some optimization work on specific problems.

### THE ROUTH-HURWITZ CRITERION

The Routh-Hurwitz criterion is a very powerful method for determining the stability of a linear control system by determining the number of roots of the system's characteristic equation in each half of the complex plane. As the system characteristic equation is a polynomial of order  $n$  and of real coefficients, this procedure turns out to consist of several rules applied to the coefficients of this polynomial in order to find the number of roots

in each half of the complex-plane without solving for the exact location of these roots.

The first step in the application of the Routh criterion is to form the Routh array from the coefficients of the polynomial:

$$a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \dots + a_2 x^2 + a_1 x + a_0 = 0$$

as follows:

$n$	$a_n$	$a_{n-2}$	$a_{n-4}$	$\dots$
$n - 1$	$a_{n-1}$	$a_{n-3}$	$a_{n-5}$	$\dots$
$n - 2$	$c_1$	$c_2$	$c_3$	$\dots$
$n - 3$	$d_1$	$d_2$	$d_3$	$\dots$
$\dots$	$\dots$	$\dots$	$\dots$	
$\dots$	$\dots$	$\dots$	$\dots$	
2	$k_1$	$k_2$		
1	$l_1$	$l_2$		
0	$m_1$			

where:

$$c_j = \frac{a_{n-1}a_{n-2j} - a_n a_{n-2j-1}}{a_{n-1}} \quad j = 1, 2, 3, \dots$$

The  $d$ 's are also determined similarly to  $c$ 's by using the second and third rows. The procedure is continued down to the zero row. The Routh criterion may be stated as follows: the number of roots of the polynomial with positive real parts is equal to the number of sign changes in the first column of the Routh array. To illustrate the Routh table, the polynomial  $P(x) = 2x^6 + 4x^5 + 2x^4 - x^3 + 2x - 2$  [4] is taken. The Routh table of this polynomial will be:

To determine the roots of a polynomial, the  $y$ -axis is shifted to the left until all roots of the shifted polynomial lie to the right of the shifted axis. Progressive shifts of the  $y$ -axis to the right by an amount  $C$  are then taken and a count of the number of roots in the right plane is obtained after each shift. Once the axis shift produces a change in the number of polynomial right-half plane roots, it means that some roots have been reached. Then a special procedure is used to determine these roots.

$X^6$	2	2	0	-2
$X^5$	4	-1	2	0
$X^4$	$\frac{(4)(2) - (2)(-1)}{4} = 2.5$	$\frac{(4)(0) - (2)(2)}{4} = -1$	$\frac{(4)(-2) - (2)(0)}{4} = -2$	0
$X^3$	$\frac{(2.5)(-1) - (4)(-1)}{2.5} = 0.6$	$\frac{(2.5)(2) - (4)(-2)}{2.5} = 5.2$	0	0
$X^2$	$\frac{(0.6)(-1) - (2.5)(5.2)}{0.6} = -22.67$	$\frac{(0.6)(-2) - (2.5)(0)}{0.6} = -2$	0	0
$X^1$	$\frac{(-22.67)(5.2) - (0.6)(-2)}{-22.67} = 5.142$	0	0	0
$X^0$	$\frac{(5.142)(-2) - (-22.67)(0)}{5.142} = -2$	0	0	0

	A	B	C	D	E	F	G	H	I	J	K
1	0	3	0.000105								
2	0	-3	1	2	11	3	1				
3	1	10	1	2	12	2					
4		8	1	233	4976	17688	5935		0	8	
5		7	-23	-1357	-11793	-15379			1		
6		6	174	4463.26087	17019.35	5935			1		
7		5	-767.02674	-9543.31609	-14594.5	0			1		
8		4	2298.3654	13708.597	5935				1		
9		3	-4968.3744	-12613.8133					1		
10		2	7873.4586	5935					1		
11		1	-8868.6609						1		
12		0	5935						1		
13		0							8	0	
14			REAL	COMPLEX	ROOTS						
15			0	0	0						
16			0	0	0						
17			0	0	0						
18			0	0	0						
19			0	0	0						
20			0	0	0						
21			0	0	0						
22			0	0	0						
23											
24											

Fig. 1. Spreadsheet layout of the Routh shifting method.

### SPREADSHEET PROGRAM FOR THE ROUTH-HURWITZ CRITERION

The first step in this work is to compute and to display the Routh array of a given polynomial. This can be achieved using the procedure outlined in the previous section. To illustrate the procedure, the polynomial:

$$p(x) = x^8 + x^7 + 2x^6 + 2x^5 + 11x^4 + 12x^3 + 3x^2 + 2x + 1$$

is considered. As shown in Fig. 1, two tables (C2:J12) are formed one for computing the Routh coefficients (C2:G12) and the other (I4:J12) for computing the number of roots in each half of the complex plane.

The first table also consists of two parts. The upper part (C2:G3) displays the original coefficients of the polynomial, and the lower part (C4:G12) displays the shifted coefficients and the Routh array. When the shift is zero, rows 4 and 5 will be the same as rows 2 and 3. When the shift is a given constant ( $C = 3$  in this case), the new coefficients are computed by simply replacing  $x$  in the original polynomial by  $(x - C)$ . These new coefficients are obtained recursively using the formula:

$$b_{n-k} = \sum_{i=0}^k a_{n-i} \binom{n-i}{k-i} C^{k-i} \quad k = 0, 1, \dots, n \quad (1)$$

In equation (1), are the shifted coefficients and are the original coefficients. The second table consists

of two columns (Column I and J). Column I computes the number of sign changes in the first column of the Routh table. This can be done by multiplying the corresponding row coefficient of the first column of the Routh table with the previous row first column value. This multiplication is negative if and only if there is a change of sign. The formula used in cell I6 in this case is given by:

$$= \text{IF}(C6 * C5 > 0; 0; 1) \quad (2)$$

This formula is copied to the other cells of column I. After the initial shift, the number of roots to the right of the shifted axis is determined by the number of sign changes, i.e. the number of 1's in the I column of Fig. 1. This number is obtained in Cell I13.

$$= \text{SUM}(I4, I12) \quad (3)$$

The result obtained in Cell I13 is copied in Cell J4. Due to the row-wise calculation available in Excel, the difference between the previous value (Cell J4) and the current value (I13) is obtained in Cell J13. This index displays the number of roots crossed after each iteration is taken. In cell B13, a difference between the order of the polynomial, 8 in our case, and the number displayed in I13 is taken. This value will inform the user, once incremented, of the number of roots that they have been determined. Note that the size of the starting shift is entered by the user in Cell B1. This number should always be large enough so in this example, the number 8 is displayed in Cell I13, or 0 in cell B13; that is all roots of the polynomial are to the right of the shift. The user

enters in Cell C1 the step size or the precision value. Counter is introduced in Cell A2 using the following formula:

$$= \text{IF}(\$A\$1 = 0; 0; A2 + C1) \quad (4)$$

As can be concluded from equation (4), A1 is an initialization flag, C1 is the increment, (C1 + A2) is the shift after each iteration, and the shift  $(-B1 + A2)$  is displayed in cell B2.

The roots of the polynomial in question are displayed in a 3-column table below the Routh array as shown in Fig. 1. The first column of this table displays the simple real roots, the second column displays the real part of the complex roots, and the third column displays the complex part of the complex roots. It will also display real double roots for simplicity reason as discussed later. At start and once a correct starting shift is entered, the values of the Cells B13 and J13 are zeros. This indicates that all roots of the polynomial are to the right of the shift introduced in Cell B1. When a single real root is crossed, two things will take place. The value of Cell J13 will be equal to 1 and the value displayed in Cell B13 will also be incremented by 1. In order to display this root, the following formula is entered in Cell B15 and copied in the remaining cells of the first column of the root table taking into consideration the pattern of the root.

$$= \text{IF}(\$A\$1 = 1; \text{IF}(\text{AND}(\text{MOD}(\$J\$13; 2) = 1; B15 = 0; B16 <> 0); \$B\$2; B15); 0) \quad (5)$$

The above formula states that if the flag is 1 and one root is crossed, the location of the shift axis B2 is displayed. Using the MOD formula in equation (5) has also the advantage of determining triple roots, and roots of order 5.

When double roots are crossed, the index in cell J13 will be incremented by 2. In this case, either these roots are real or complex conjugate roots. To make the problem much easier, these roots are displayed in the complex roots section with 0 in the complex part when they are real, and similarly 0 in the real part when they are purely imaginary roots. The complex part of the complex-conjugate roots is obtained from the auxiliary polynomial formed from the  $s^2$  row of the Routh table. In this case, the real part will be equal to the shift value. It is determined using the following spreadsheet formula:

$$= \text{IF}(\$A\$1 = 1; \text{IF}(\text{AND}(\$J\$13 = 2; I13 = 6); \$B\$2; C15); 0) \quad (6)$$

Where the imaginary part is found to be:

$$= \text{IF}(\$A\$1 = 1; \text{IF}(\text{AND}(\$J\$13 = 2; \$I\$13 = 6); (\$D\$10/\$C\$10)^{0.5}; D15); 0) \quad (7)$$

It is to note that the negative sign for the complex conjugate roots is not introduced in equation (7) for clarity.

## THE BAIRSTOW'S METHOD

As indicated earlier, this method is based on dividing the original  $n$ th order polynomial  $f(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n$  by a quadratic term of the form  $(x^2 - rx - s)$ . The result is a polynomial  $g(x) = b_2 + b_3x + \dots + b_nx^{n-2}$  of order  $(n - 2)$  and a remainder  $R = b_1(x - r) + b_0$ . Simple recurrence relationships can be used for determining the coefficients of  $g(x)$ . These are found to be:

$$\begin{aligned} b_n &= a_n \\ b_{n-1} &= a_{n-1} + rb_n \\ b_i &= a_i + rb_{i+1} + sb_{i+2} \quad \text{for } i = 0, 1, \dots, n-2 \end{aligned} \quad (8)$$

From (8), we may obtain:

$$\begin{aligned} b_0 &= a_0 + rb_1 + sb_2 \\ b_1 &= a_1 + rb_2 + sb_3 \end{aligned} \quad (9)$$

Because  $b_0$  and  $b_1$  are both functions of  $r$  and  $s$ , the above equations can be expanded using a Taylor series leading when neglecting the high-order term to:

$$\begin{aligned} b_1(r + \Delta r, s + \Delta s) &\approx b_1 + \frac{\partial b_1}{\partial r} \Delta r + \frac{\partial b_1}{\partial s} \Delta s \\ b_0(r + \Delta r, s + \Delta s) &\approx b_0 + \frac{\partial b_0}{\partial r} \Delta r + \frac{\partial b_0}{\partial s} \Delta s \end{aligned} \quad (10)$$

Bairstow showed that the partial derivatives in equation (10) can be obtained by a synthetic division of the  $b$ 's in a fashion similar to the way the  $b$ 's were derived.

$$\begin{aligned} c_n &= b_n \\ c_{n-1} &= b_{n-1} + rc_n \\ c_i &= b_i + rc_{i+1} + sc_{i+2} \quad \text{for } i = 1, \dots, n-2 \end{aligned} \quad (11)$$

Using equation (11), and the fact that  $\Delta r$  and  $\Delta s$  are very small, equation (10) becomes:

$$\begin{aligned} -b_1 &= c_2 \Delta r + c_3 \Delta s \\ -b_0 &= c_1 \Delta r + c_2 \Delta s \end{aligned} \quad (12)$$

These two equations in (12) are solved for  $\Delta r$  and  $\Delta s$ , and the obtained results are used to improve

	A	B	C	D	E	F	G	H	I	J
1	<b>BAIRSTOW'S METHOD</b>									
2										
3	<b>Initial Guesses</b>			<b>r &amp; s After Iterating</b>			<b>Order of Polynomial</b>		<b>Max Error</b>	
4	r	0		r	0		Pol. Order	8		0.000001
5	s	0		s	0		New Order	9		
6										
7	Input Pol.	1	0	2	2	-11	-13	3	2	1
8										
9	f(x)	0	0	0	0	0	0	0	0	0
10	b	0	0	0	0	0	0	0	0	0
11	c	0	0	0	0	0	0	0	0	0
12										
13	<b>Solving For New Increments of r &amp; s</b>									
14	Equ 1	0	0	0						
15	Equ 2	0	0	0						
16		Delta r	0							
17		Delta s	0							
18										
19	Error	0	0							
20										
21	<b>Roots</b>									
22	Real	0	0	0	0	0	0	0	0	0
23	Imaginary	0	0	0	0	0	0	0	0	0

Fig. 2. Spreadsheet layout of the Bairstow's method.

initial guess. At each step, an approximate error in r and s can be found as:

$$|\epsilon_r| = \left| \frac{\Delta r}{r} \right| \times 100\% \quad (13)$$

$$|\epsilon_s| = \left| \frac{\Delta s}{s} \right| \times 100\%$$

When both estimates are less than the precision value, the roots are then found to be:

$$x = \frac{r \pm \sqrt{r^2 + 4s}}{2} \quad (14)$$

and the procedure continues for the polynomial g(x). If g(x) is a first-order polynomial, the single real root can be evaluated simply as  $x = -s/r$ .

#### SPREADSHEET IMPLEMENTATION OF THE BAIRSTOW'S METHOD

The spreadsheet program of Bairstow's method is shown in Fig. 2. In this figure, the coefficients of the polynomial are entered in row 7, the initial guess for r and s are entered in cells B4 and B5, the order of the polynomial is entered in cell H4, and the maximum error is entered in cell J4. After each iteration, the coefficients b's and c's defined in equations (8) and (11) are computed in rows 10 and 11. For the b's coefficients the following formulas are used:

$$B10 = B9$$

$$C10 = C9 + E4 * B10 \quad (15)$$

$$D10 = D9 + \$E\$4 * C10 + \$E\$5 * B10$$

In equations (15), E4 and E5 represent the new values of r and s obtained after computing  $\Delta r$  and  $\Delta s$  as shown later, and the third formula is copied for the remaining elements of the row. Similarly, the c's coefficients are obtained using the following formulas:

$$B11 = B10$$

$$C11 = C10 + E4 * B11 \quad (16)$$

$$D11 = D10 + \$E\$4 * C11 + \$E\$5 * B11$$

The third formula in equation (16) is copied for the remaining elements of this row. The next step is to solve for the new increment of r and s. This is done using equation (12) and implemented in rows 14, 15, 16, and 17 as shown in Fig 2.

The first equation of (12) is first obtained using the following formulas:

$$c_2 = B14 = \text{IF}(H5 < 3; 0; \text{INDEX}(B11 : P11; B11; H5 - 1))$$

$$c_3 = C14 = \text{IF}(H5 < 3; 0; \text{INDEX}(B11 : P11; B11; H5 - 2)) \quad (17)$$

$$-b_1 = D14 = \text{IF}(H5 < 3; 0; -\text{INDEX}(B10 : P10; B10; H5))$$

In equation (17), the index function is used whose objective is to return the value specified with the given array. In a similar fashion, the coefficients of the second equation of (12) are found in cells B15,

	A	B	C	D	E	F	G	H	I	J	K
1	1	3	0.000105								
2	4.2	1.2	1	2	11	3	1				
3	1	10	0	2	-13	2					
4		6	1	42.32	211.352	231.6156	23.31402		0	0	
5		7	9.6	113.168	277.0659	108.3333			0		
6		6	30.531667	182.490967	220.3308	23.31402			0		
7		5	55.787795	207.787815	101.0027	0			0		
8		4	68.772397	165.053849	23.31402				0		
9		3	73.897028	82.0905019					0		
10		2	88.656181	23.314025					0		
11		1	62.65771						0		
12		0	23.314025						0		
13		8							0	0	
14			REAL	COMPLEX	ROOTS						
15		0	-1.371555	1.33744785							
16		0	-1.371555	1.33744785							
17		0	-0.210465	0.25265857							
18		0	-0.210465	0.25265857							
19		0	0.692115	0.39097488							
20		0	0.692115	0.39097488							
21		0	0.89025	1.6924194							
22		0	0.89025	1.6924194							
23											

Fig. 3. Roots of  $x^8 + 2x^6 + 2x^5 + 11x^4 - 13x^3 + 3x^2 + 2x + 1 = 0$  using Routh procedure.

C15, and D15 respectively. These spreadsheet formulas are:

$$c_1 = B15 = \text{IF}(H5 < 3; 0; \text{INDEX}(B11 : P11; B11; H5))$$

$$c_2 = C15 = B14 \quad (18)$$

$$-b_0 = \text{IF}(H5 < 3; 0; -\text{INDEX}(B10 : P10; B10; H5 + 1))$$

Once these coefficients are found, now we can solve for  $\Delta r$  and  $\Delta s$  using the following formulas:

$$\Delta r = C16 = \text{IF}((B14 * C15 - C14 * B15) = 0; 0; -(C14 * D15 - D14 * C15) \div (B14 * C15 - C14 * B15))$$

$$\Delta s = C17 = \text{IF}((B14 * C15 - C14 * B15) = 0; 0; (B14 * D15 - D14 * B15) \div (B14 * C15 - C14 * B15))$$

After each iteration, the new values of  $r$  and  $s$  are found to be:

$$\text{New } r = E4 = \text{IF}(\text{OR}(E4 = 0; B4 = 0); B4; \text{IF}(H5 = 2; -C10; E4 + C16))$$

$$\text{New } s = E5 = \text{IF}(\text{OR}(E5 = 0; B5 = 0); B5; \text{IF}(H5 = 2; -D10; E5 + C17)) \quad (20)$$

Similarly, the new order of the polynomial is:

$$H5 = \text{IF}(E4 = B4; H4; \text{IF}(H5 = 0; 0; \text{IF}(H5 = 1; 1; \text{IF}(\text{AND}(B19 \leq J4; C19 \leq J4); H5 - 2; H5)))) \quad (21)$$

Finally and once the error computed in row 19 is less than the value entered in J4, the real part and the imaginary part of each root of the polynomial  $f(x)$  are found in rows 22 and 23 using the following spreadsheet formulas:

$$\text{real} = \text{IF}(\text{AND}(\$B\$4 = 0; \$B\$5 = 0); 0; \text{IF}(\$H\$4 = 1; -\$C\$10/\$B\$10; \text{IF}(\$H\$4 = 2; \text{IF}((\$E\$4^2 + 4 * \$E\$5) < 0; \$E\$4/2; (\$E\$4 + \text{SQRT}(\$E\$4^2 + 4 * \$E\$5))/2); \text{IF}((\$H\$4 - \$H\$5) < > 0; B22; \text{IF}((\$E\$4^2 + 4 * \$E\$5) < 0; \$E\$4/2; (\$E\$4 + \text{SQRT}(\$E\$4^2 + 4 * \$E\$5))/2))))))$$

$$\text{Imag} = \text{IF}(\text{AND}(\$B\$4 = 0; \$B\$5 = 0); 0; \text{IF}(\$H\$4 = 1; 0; \text{IF}(\$H\$4 = 2; \text{IF}((\$E\$4^2 + 4 * \$E\$5) > 0; 0; \text{SQRT}(\text{ABS}(\$E\$4^2 + 4 * \$E\$5))/2); \text{IF}((\$H\$4 - \$H\$5) < > 0; B23; \text{IF}((\$E\$4^2 + 4 * \$E\$5) > 0; 0; \text{SQRT}(\text{ABS}(\$E\$4^2 + 4 * \$E\$5))/2)))))) \quad (22)$$

After the roots are found, the process is continued

	A	B	C	D	E	F	G	H	I	J
1	BAIRSTOW'S METHOD									
2										
3	Initial Guesses		r & s After Iterating			Order of Polynomial		Max Error		
4	r	2		r	1.780323		Pol. Order	8		0.000001
5	s	-2		s	-3.98497		New Order	0		
6										
7	Input Pol.	1	0	2	2	11	-13	3	2	1
8										
9	f(x)	1	0.962973	2.770062	4.399655	14.62103	-8.6E-11	1.52E-10	1.12E-07	-1.2919E-06
10	b	1	2.743296	3.669048	0	0	-8.6E-11	-8.6E-13	1.12E-07	-1.0922E-06
11	c	1	4.523619	7.737584	-4.25108	-38.4023	-51.4281	61.47328	314.3816	314.7317349
12										
13	Solving For New Increments of r & s									
14	Equ 1	0	0	0						
15	Equ 2	0	0	0						
16		Delta r	0							
17		Delta s	0							
18										
19	Error	0	0							
20										
21	Roots									
22	Real	0.692026	0.692026	-0.21054	-0.21054	-1.37165	-1.37165	0.890161	0.890161	0
23	Imaginary	0.391021	-0.39102	0.252838	-0.25284	1.337023	-1.33702	1.786779	-1.78678	0

Fig. 4. Roots of  $x^8 + 2x^6 + 2x^5 + 11x^4 - 13x^3 + 3x^2 + 2x + 1 = 0$  using Bairstow's procedure.

by replacing the values in row 9 with new values calculated by the following formula:

= IF(AND(\$B\$4 = 0; \$B\$5 = 0); 0;  
 IF(AND(\$E\$4 = \$B\$4; \$E\$5 = \$B\$5); B7;  
 IF(AND(\$B\$19 > \$J\$4; \$C\$19 > \$J\$4); B9;  
 IF(\$H\$5 < 3; B9; B10))) (23)

The above formula is copied for the remaining cells of row 9.

## EXAMPLES AND COMPARISON

### Example 1

In order to check the program, the following polynomial is considered:

$$x^8 + 2x^6 + 2x^5 + 11x^4 - 13x^3 + 3x^2 + 2x + 1 = 0$$

The starting shift value is set to 3, and the precision value or the step size is set to 0.000105. The roots of this polynomial are given in Fig. 3.

Using Matlab, the following roots are

	A	B	C	D	E	F	G	H	I	J	K
1	1	120	0.000105								
2	120.5708	0.570765	1	1	-2	3	1				
3		10	118	2	-3	2					
4		8	1	481.573525	783.9539	150.3351	4.844296	0	0		
5		7	122.56612	823.101924	444.3683	31.14546		0			
6		6	474.85795	780.328391	150.081	4.844296		0			
7		5	621.69049	405.630683	29.8951	0		0			
8		4	470.50066	127.246626	4.844296			0			
9		3	237.49486	23.4941421				0			
10		2	80.702421	4.8442963				0			
11		1	9.2381205					0			
12		0	4.8442963					0			
13		8						0	0		
14		REAL	COMPLEX	ROOTS							
15		-117.9917	0	0							
16		-0.50307	0	0							
17		0	-0.258735	0.32233962							
18		0	-0.258735	0.32233962							
19		0	-0.004845	0.54285088							
20		0	-0.004845	0.54285088							
21		0	0.511125	0.27043639							
22		0	0.511125	0.27043639							
23											

Fig. 5. Roots of  $x^8 + 118x^7 + x^6 + 2x^5 - 2x^4 - 3x^3 + 3x^2 + 2x + 1 = 0$  using Routh procedure.

A	B	C	D	E	F	G	H	I	J	K
BAIRSTOW'S METHOD										
Initial Guesses		r & s After Iterating		Order of Polynomial				Max Error		
r	2	r	-1.47826	Pol. Order		9	0.000001			
s	3	s	-1.48854	New Order		1				
Input Pol.	1	-2	3	0	5	-4	7	8	9	3
f(x)	1	1.887456	2.093432	0.609101	8.16E-10	-3.9E-11	-8.2E-10	8.53E-10	-4.9151E-07	-4.7E-06
h	1	0.409195	9.57E-14	-2.1E-13	8.16E-10	-1.2E-09	-2E-10	3E-09	-4.9665E-07	-3.9E-06
c	1	-1.06907	0.091824	1.455604	-2.28845	1.216201	1.608569	-4.18824	3.796904314	0.62153
Solving For New Increments of r & s										
Equ 1	0	0	0							
Equ 2	0	0	0							
	Delta r	0								
	Delta s	0								
Error	0	0								
Roots										
Real	1.251229	1.251229	1.05372	1.05372	-0.36122	-0.36122	-0.73913	-0.73913	-0.40919469	0
Imaginary	1.099346	-1.09935	1.344496	-1.3445	0.691348	-0.69135	0.970681	-0.97068	0	0

Fig. 6. Roots of  $x^9 - 2x^8 + 3x^7 + 5x^5 - 4x^4 + 7x^3 + 8x^2 + 9x + 3 = 0$  using Bairstow's procedure.

obtained:  $0.8902 \pm 1.7868j$ ,  $-1.3716 \pm 1.3370j$ ,  $0.6920 \pm 0.3910j$ , and  $-0.2105 \pm 0.2528j$ . The result of the same polynomial using the Bairstow's method is shown in Fig. 4.

As concluded, the two procedures give, with slight approximation, the accurate values.

#### Example 2: polynomial with a high coefficient

In this case, let us consider the following polynomial:

$$x^8 + 118x^7 + x^6 + 2x^5 - 2x^4 - 3x^3 + 3x^2 + 2x + 1 = 0$$

As we can see, the  $x^7$  coefficient is very large as compared to others. The result obtained using this procedure is given in Fig. 5.

In computing the roots using Matlab, the following result is obtained:  $1.0e + 002 * [-1.1799, 0.0051 \pm 0.0027j, -0.0000 \pm 0.0054j, -0.0050, -0.0026 \pm 0.0032j]$ . Clearly from the above, that for this case Matlab did not display

the real part of the complex roots as considered to be very small comparing to other roots. Using Bairstow's method, the results are shown in Fig. 6 for a polynomial of order 9. Results are the same as for the Matlab.

## CONCLUSION

The outlined spreadsheet solutions are capable of determining the roots of a polynomial of any degree. They have been tried for a polynomial with degree 8 with high success and the results were very much comparable with already existing tools. The user will simply enter the coefficients of the polynomial in the indicated cells, and starting point, and the error size. The program, with few iterations, will then display the roots of the polynomial within the error specified by the user. The procedure is easy and straightforward and can be used for other applications.

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