Numerical and Experimental Analysis of a Cantilever Beam: a Laboratory Project to Introduce Geometric Nonlinearity in Mechanics of Materials*

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The classical problem of deflection of a cantilever beam of linear elastic material, under the action of a uniformly distributed load along its length (its own weight) and an external vertical concentrated load at the free end, is experimentally and numerically analysed. We present the differential equation governing the behaviour of this system and show that this equation, although straightforward in appearance, is in fact rather difficult to solve due to the presence of a nonlinear term. The experiment described in this paper is an easy way to introduce students to the concept of geometric nonlinearity in mechanics of materials. The ANSYS program is used to numerically evaluate the system and calculate Young's modulus of the beam material. Finally, we compare the numerical results with the experimental ones obtained in the laboratory.

AUTHOR'S QUESTIONNAIRE

1. This paper proposes the introduction of the concept of geometric nonlinearity in an intermediate mechanics of materials course by means of the analysis of a simple laboratory experiment on the bending of a cantilever beam.
2. The experimental set-up is composed of very simple elements and only easy experimental measurements—lengths and masses—need be made.
3. The differential equation governing the behaviour of this system is derived without difficulty and by analysing this equation it is possible to show that, although straightforward in appearance, it is in fact rather difficult to solve due to the presence of a nonlinear term.
4. The numerical analysis of the beam is made using a personal computer with the help of the ANSYS program, and the way in which the modulus of elasticity of the beam material can be obtained is very illustrative for students.
5. The behaviour of the cantilever beam experimentally analysed is nonlinear except for an external load $F = 0$. This enables students to understand when the linear theory, that is a first order approximation of the general case, can be applied and when it is necessary to consider the more general nonlinear theory.
6. The experiment described in this paper provides students with an understanding of geometric nonlinearity and also a better understanding of the basic concepts of mechanics of materials. Additionally, the experiment enables students to apply these well-understood concepts to a practical problem.

INTRODUCTION

BEAMS are common elements of many architectural, civil and mechanical engineering structures [1–5] and the study of the bending of straight beams forms an important and essential part of the study of the broad field of mechanics of materials and structural mechanics. All undergraduate courses on these topics include the analysis of the bending of beams, but only small deflections of the beam are usually considered. In such as case, the differential equation that governs the behaviour of the beam is linear and can be easily solved.

However, we believe that the motivation of students can be enhanced if some of the problems analysed in more specialised books on mechanics of materials or structural mechanics are included in the undergraduate courses on these topics. However, it is evident that these advanced problems cannot be presented to undergraduate students in the same way as is done in the specialised monographs [6]. As we will discuss in this paper, it is possible to introduce the concept of geometric nonlinearity by studying a very simple

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experiment that students can easily analyse in the laboratory, that is, the bending of a cantilever beam. The mathematical treatment of the equilibrium of this system does not involve great difficulty [5]. Nevertheless, unless small deflections are considered, an analytical solution does not exist, since for large deflections a differential equation with a nonlinear term must be solved. The problem is said to involve geometric nonlinearity [6–9].

The purpose of this paper is to analyse a simple laboratory experiment in order to introduce the concept of geometric nonlinearity in a course on mechanics or strength of materials. This type of nonlinearity is related to the nonlinear behaviour of deformable bodies, such as beams, plates and shells, when the relationship between the extensional strains and shear strains, on the one hand, and the displacement, on the other, is taken to be nonlinear, resulting in nonlinear strain-displacement relations. As a consequence of this fact, the differential equations governing this system will turn out to be nonlinear. This is true in spite of the fact that the relationship between curvatures and displacements is assumed to be linear. The experiment will allow students to explore the deflections of a loaded cantilever beam and to observe in a simple way the nonlinear behaviour of the beam.

We will consider a geometrically nonlinear beam problem by numerically and experimentally analysing the large deflections of a cantilever beam of linear elastic material, under the action of an external vertical tip load at the free end and a uniformly distributed load along its length (its own weight). Under the action of these external loads, the beam deflects into a curve called the elastic curve or elastica. If the thickness of the cantilever is small compared to its length, the theory of elastica adequately describes the large, nonlinear displacements due to the external loads.

The experimental analysis is completed with a numerical analysis of the system using the ANSYS program, a comprehensive finite element package, which enables students to solve the nonlinear differential equation and to obtain the modulus of elasticity of the beam material. To do this, students must fit the experimental results of the vertical displacement at the free end to the numerically calculated values for different values of the modulus of elasticity or Young's modulus by minimising the sum of the mean square root. Using the modulus of elasticity previously obtained, and with the help of the ANSYS program [10], students can obtain the elastic curves of the cantilever beam for different external loads and compare these curves with the experimental ones. ANSYS is a finite element modelling and analysis tool. It can be used to analyse complex problems in mechanical structures, thermal processes, computational fluid dynamics, magnetics, electrical fields, just to mention some of its applications. ANSYS provides a rich graphics capability that can be used to display results of analysis on a high-resolution graphics workstation.

In recent years, personal computers have become everyday tools of engineering students, who are familiar with programs such as Mathematica, Matlab or ANSYS, which also have student versions. Thanks to the use of computers and commercial software, students can now gain additional insight into fundamental concepts by numerical experimentation and visualisation and are able to solve more complex problems. Use of the ANSYS program to carry out numerical experiments in mechanics of materials has been analysed, for example, by Moaveni [11]. In addition, since 1998 in the Miguel Hernández University (Spain), one of the authors of this paper (T.B.) has proposed numerical experiments using the ANSYS program for students of an introductory level course on strength of materials (second-year), an advanced mechanics of materials course for materials engineers (fourth-year) and a structural analysis course (fifth-year), in a similar way to that described in Moaveni's paper. However, in this study we supplement the numerical simulations performed using the ANSYS program with laboratory experiments, thereby providing the students with a more comprehensive view of the problem analysed.

Fig. 1. Photograph of a cantilever beam loaded with an external vertical concentrated load at the free end and a distributed load (its own weight).
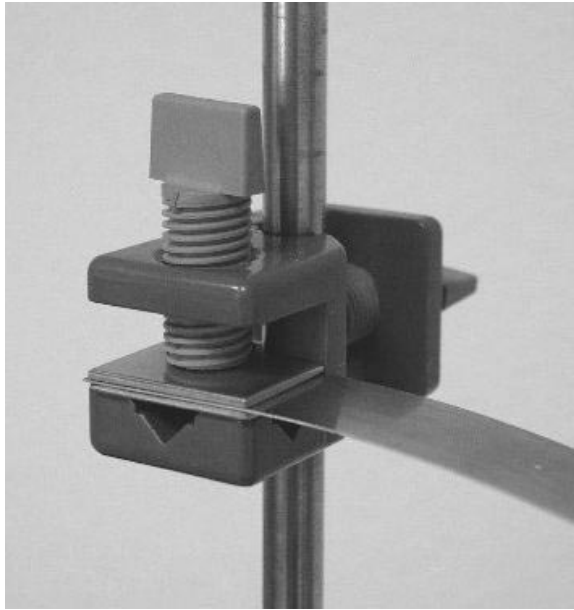
Fig. 2. Photograph of the fixation of the beam to a vertical stand rod by means of a multi-clamp using two small metallic pieces.

Fig. 3. Experimental measurement of the elastica of the cantilever beam as well as of the vertical displacement of the free end, δ_y.

EXPERIMENTAL SET-UP

In the laboratory it is possible to design simple experiments in order to analyse the deflection of a cantilever beam with a tip load applied at the free end. For example, Fig. 1 shows a photograph of a system made up of a flexible steel beam of rectangular cross-section built-in at one end and loaded at the free end with a mass. The beam is fixed to a vertical stand rod by means of a multi-clamp using two small metallic pieces, which provide a better support (Fig. 2). The length of the beam is $L = 0.40$ m and it has a uniform rectangular cross-section of width $b = 0.025$ m and height $h = 0.0004$ m. The weight of the beam and the value of the load uniformly distributed over its entire length are $W = 0.3032$ N and $w = W/L = 0.758$ N/m, respectively.

With this experimental set-up the students can, for instance, determine the vertical deflection of the end free as a function of the applied load, or the shape the beam adopts under the action of that force, by using vertical and horizontal rulers as can be seen in Fig. 3. This figure shows the procedure followed to obtain the experimental measurements of the elastic curve of the beam as well as of the horizontal and vertical displacements at the free end. The students can relate these measurements to geometric parameters of the beam (its length and the moment of inertia of its rectangular cross-section), as well as to the material of which it is made (using Young's modulus). This system is made up of very simple elements and only easy experimental measurements (basically lengths and masses) need be made. In addition, mathematical treatment of the equilibrium of the system does not involve great difficulty [12], however for large deflections a differential equation with a nonlinear term must be solved [8].
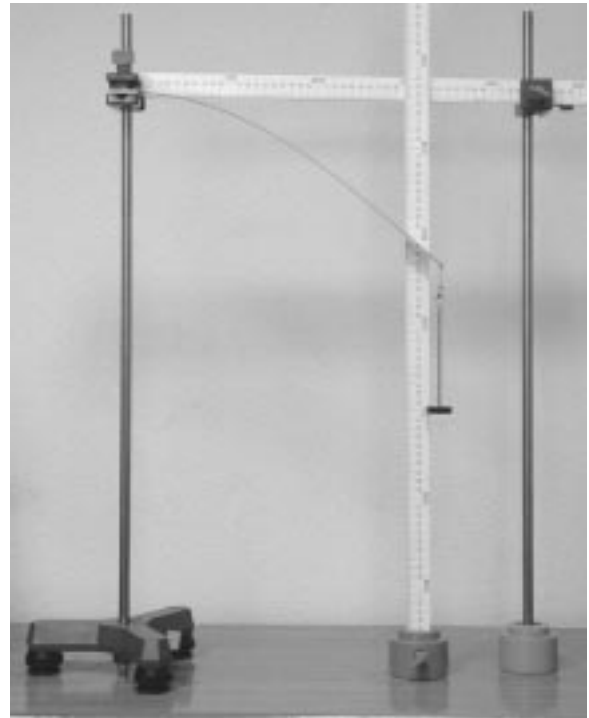
THEORETICAL ANALYSIS

Let us consider the case of a long, thin, cantilever beam of uniform rectangular cross section made of a linear elastic material, whose weight is W, subjected to a tip load F as shown in Figs 4 and 5. In this study, we assume that the beam is non-extensible and the strains remain small. Firstly, we assume that Bernoulli-Euler's hypothesis is valid, that is, plane cross-sections which are perpendicular to the neutral axis before deformation remain plane and perpendicular to the neutral axis after deformation. Next, we also assume that the plane-sections do not change their shape or area.

The Bernoulli-Euler bending moment-curvature relationship for a uniform-section rectangular beam of linear elastic material can be written as follows:

$$M = EI\kappa \qquad (1)$$

where E is the Young's modulus of the material, M and κ are the bending moment and the curvature at any point of the beam, respectively, and I is the moment of inertia (the second moment of area) of the beam cross-section about the neutral axis [3–6]. The product EI, which depends on the type of material and the geometrical characteristics of the cross-section of the beam, is known as the flexural rigidity.
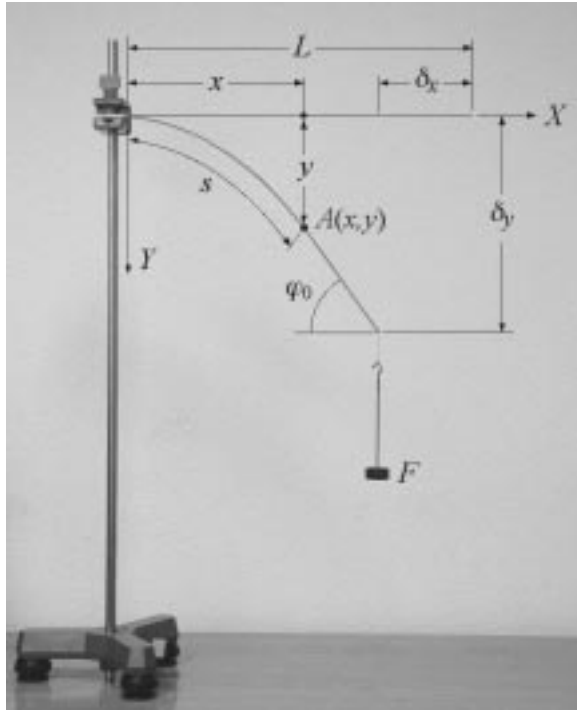
Fig. 4. Photograph of the cantilever beam under the action of a uniformly distributed load and a vertical concentrated load at the free end, and definition of parameters.

The moment of inertia of the cross section is derived from equation (1):

$$I = \tfrac{1}{12} b h^3 \tag{2}$$

and its value for the cantilever beam experimentally analysed is $I = 1.333 \times 10^{-13}\,\mathrm{m}^4$.

Equation (1)—which involves the bending moment, M—governs the deflections of uniform rectangular cantilever beams made of linear type
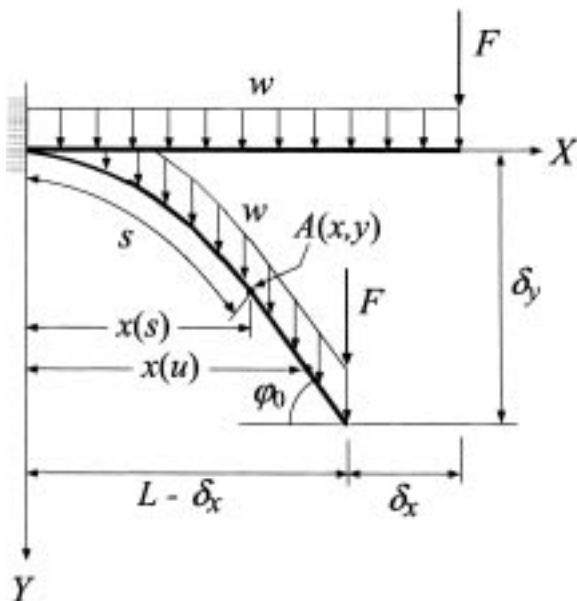

Fig. 5. Scheme of the cantilever beam under the action of a uniformly distributed load and a vertical concentrated load at the free end, and definition of parameters.

material under general loading conditions. Differentiating equation (1) once with respect to s, we can obtain the equation that governs large deflections of a uniform rectangular cantilever beam:

$$\frac{\mathrm{d}\kappa}{\mathrm{d}s} = \frac{1}{EI}\frac{\mathrm{d}M}{\mathrm{d}s} \tag{3}$$

In Figs 4 and 5, δ_x and δ_y are the horizontal and vertical displacements at the free end, respectively, and φ_0 takes into account the maximum slope of the beam. We take the origin of the Cartesian co-ordinate system at the fixed end of the beam and let (x, y) be the co-ordinates of point A, and s the arc length of the beam between the fixed end and point A. The bending moment M at a point A with Cartesian co-ordinates (x, y) can be easily obtained from a free-body diagram of the segment from $u = s$ to $u = L$ in Fig. 5 as follows:

$$M(s) = \int_s^L w[x(u) - x(s)]\,\mathrm{d}u + F(L - \delta_x - x) \tag{4}$$

where $L - \delta_x - x$ is the distance from the section of the beam at a point A to the free end where force F is applied, and u is a dummy variable of s.

Differentiating equation (4) once with respect to s and recognising that $\cos\varphi = \mathrm{d}x/\mathrm{d}s$ [8]:

$$\frac{\mathrm{d}M}{\mathrm{d}s} = -w(L - s)\cos\varphi - F\cos\varphi \tag{5}$$

Using equation (5), we can write equation (2) as follows:

$$\frac{\mathrm{d}^2\varphi(s)}{\mathrm{d}s^2} = -\frac{1}{EI}[w(L - s) + F]\cos\varphi(s) \tag{6}$$

where we have taken into account the relation between κ and φ:

$$\kappa = \frac{\mathrm{d}\varphi}{\mathrm{d}s} \tag{7}$$

Equation (6) is the nonlinear differential equation that governs the deflections of a cantilever beam made of a linear material under the action of a uniformly distributed load and a vertical concentrated load at the free end.

The boundary conditions of equation (6) are:

$$\varphi(0) = 0 \tag{8}$$

$$\left(\frac{\mathrm{d}\varphi}{\mathrm{d}s}\right)_{s=L} = 0 \tag{9}$$

Taking into account that $\cos\varphi = \mathrm{d}x/\mathrm{d}s$ and $\sin\varphi = \mathrm{d}y/\mathrm{d}s$, the x and y co-ordinates of any point of the elastic curve of the cantilever beam are found as follows:

$$x(s) = \int_0^s \cos\varphi(s)\,\mathrm{d}s \tag{10}$$

$$y(s) = \int_0^s \sin\varphi(s)\,\mathrm{d}s \tag{11}$$

From Fig. 5, it is easy to see that the horizontal and vertical displacements at the free end can be obtained from equations (10) and (11) for $s = L$:

$$\delta_x = L - x(L) \tag{12}$$

$$\delta_y = y(L) \tag{13}$$

NUMERICAL AND EXPERIMENTAL RESULTS

We shall now study the large deflections of a cantilever beam using the ANSYS program, a comprehensive finite element package. We use the ANSYS structural package that simulates both the linear and nonlinear effects of structural models in a static or a dynamic environment. Firstly we have to obtain the Young's modulus of the material. To do this, we obtain experimentally the values of the vertical displacements at the free end, δ_y, for different values of the concentrated load F applied at the free end of the beam. We consider seven values for F: 0, 0.098, 0.196, 0.294, 0.392, 0.490 and 0.588 N and we obtain the theoretical value of δ_y for different values of E around the value of $E = 200$ GPa (the typical value of Young's modulus for steel) using the ANSYS program. We obtain the value of Young's modulus E by comparing the experimentally measured displacements at the free end $\delta_{y,exp}(F_j)$, where $j = 1, 2, \ldots, J$; J being the number of different external loads F considered (in our analysis $J = 7$), with the numerically calculated displacements $\delta_y(E, F_j)$. We obtain the value of E for which the sum of the mean square root χ^2 is minimum, where χ^2 is given by the following equation:

$$\chi^2(E) = \sum_{j=1}^{J} [\delta_y(E, F_j) - \delta_{y,\exp}(F_j)]^2 \tag{14}$$

In Fig. 6 we have plotted the calculated values of χ^2 as a function of E. The value of Young's
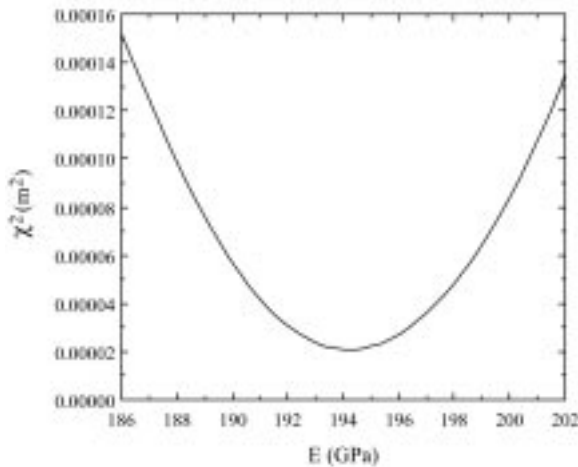
Table 1.

F (N)	δ_y (m) experimental	δ_y (m) ($E = 194.3$ GPa)	Relative error
0.000	0.089	0.0898	0.89%
0.098	0.149	0.1516	1.71%
0.196	0.195	0.1960	0.25%
0.294	0.227	0.2270	0.51%
0.392	0.251	0.2495	0.60%
0.490	0.268	0.2661	0.71%
0.588	0.281	0.2785	0.90%

modulus that minimises the quantity χ^2 is $E = 194.3$ GPa, which implies that the flexural rigidity is $EI = 0.02591$ Nm^2.

Table 1 shows the experimental values of δ_y as a function of the applied load F together with the values of δ_y calculated numerically with the aid of the ANSYS program. We also included the relative error of the values of δ_y calculated theoretically as compared with the values measured experimentally.

Figure 7 shows the displacement δ_y at the free end of the cantilever beam as a function of the external vertical concentrated load F applied at the free end. The dots correspond to the experimental data and the continuous lines to the displacements numerically calculated with the help of the ANSYS program, using the value obtained for Young's modulus, $E = 194.3$ GPa. Comparing the experimentally measured displacements with the calculated values, we can see that the agreement is satisfactory.

In Fig. 7 we have included the values of δ_y calculated, considering that the behaviour of the cantilever beam is linear (linear approximation for small deflections), using the well-known equation for the vertical displacement at the free end for a cantilever beam under the action of an external concentrated load F at the free end and a
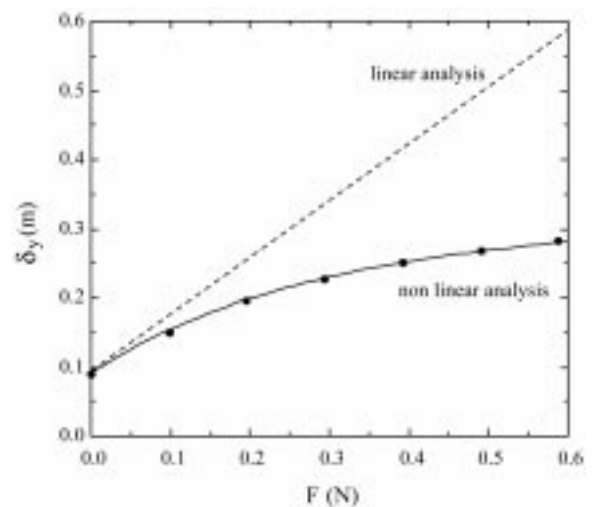

Fig. 6. Calculated values of χ^2 as a function of E.

Fig. 7. Experimentally and numerically calculated values, with $E = 194.3$ GPa, for the vertical displacement at the free end, δ_y, as a function of the concentrated load F. The discontinuous line corresponds to the values calculated using the approximative equation for small deflections (linear analysis).

Fig. 8. Photograph of the experimental elastic curve for
$F = 0.098$ N.

uniformly distributed load W along its length [5, 11]:

$$\delta_y = \frac{L^3}{24EI}(3W + 8F) \qquad (15)$$

As we can see from Fig. 7, the deflections calculated using equation (15) coincide with the experimentally measured deflections only when the load $F = 0$, whereas for all other applied loads the behaviour of the beam is clearly nonlinear. As a part of the study, students can then observe that the behaviour of the cantilever beam analysed in this experiment may be considered linear when no external load is applied ($F = 0$), as can be seen in Fig. 8. If we consider $F = 0$ in equation (15), we obtain:

$$\delta_y = \frac{WL^3}{8EI} \qquad (16)$$

Applying the approximated equation (16) for the linear case and taking into account that $\delta_y = 0.089$ m for $F = 0$ (see Table 1), we obtain $E = 204$ GPa for Young's modulus, which is higher than the value calculated using the ANSYS program.

Having calculated the value of E, we can verify that, using this value and with the aid of the ANSYS program, it is possible to obtain the elastic
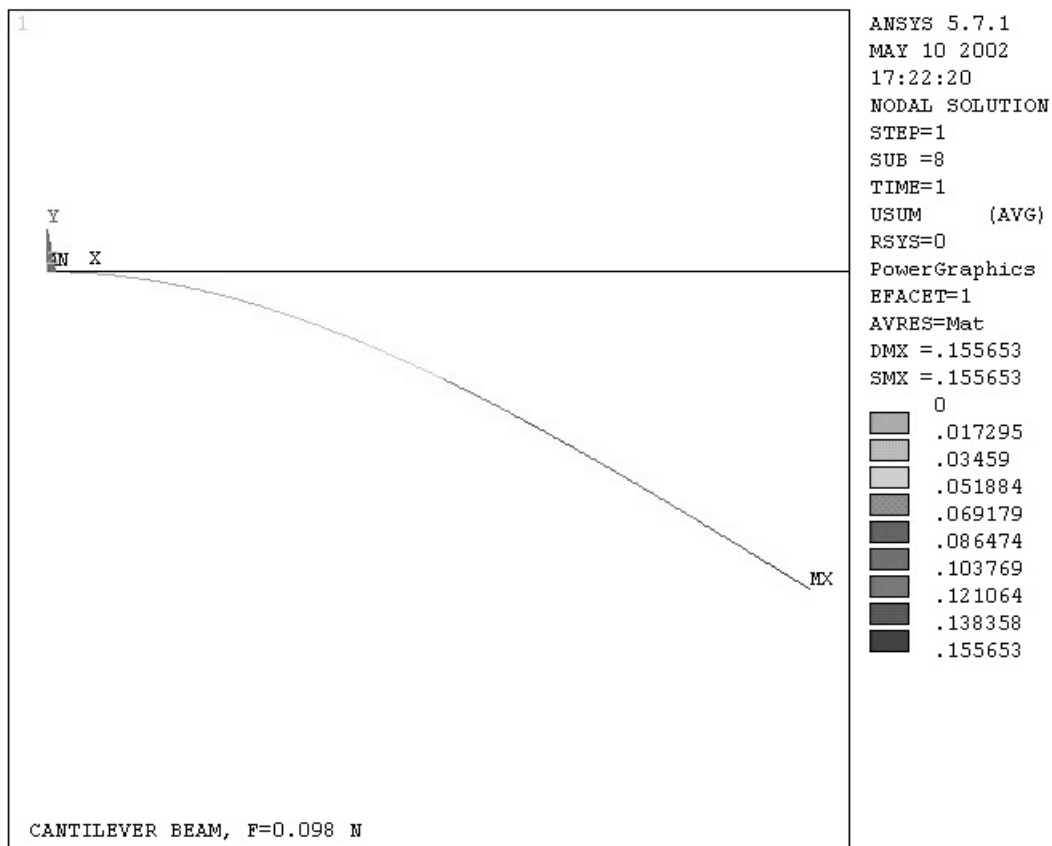

Fig. 9. Simulation of the elastic curve calculated numerically with the aid of ANSYS using $E = 194.3$ GPa for $F = 0.098$ N.
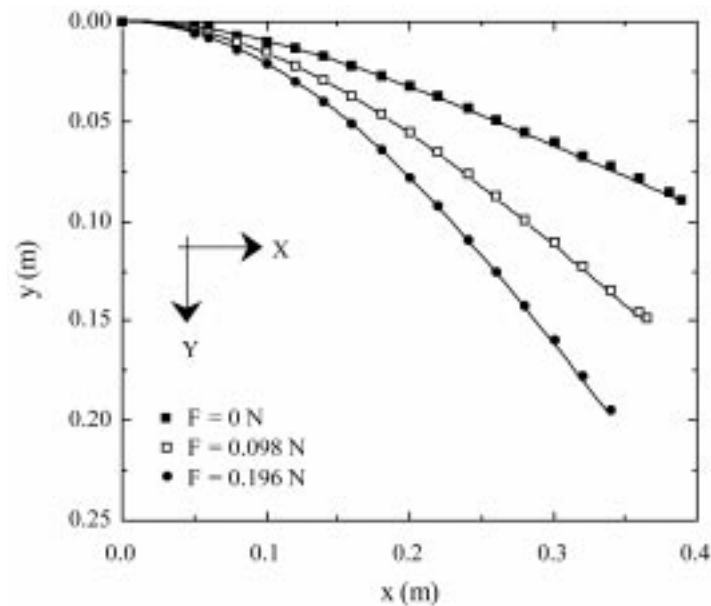
Fig. 10. Experimentally measured and numerically calculated elastic curves for $E = 194.3$ GPa.

curves for different external loads, that is, the x and y co-ordinates of the horizontal and vertical deflection at any point along the neutral axis of the cantilever beam. Figures 8 and 9 show photographs of the experimental elastic curve as well as the simulation calculated numerically with the aid of the ANSYS program, using $E = 194.3$ GPa, for the applied force $F = 0.098$ N. Finally, in Fig. 10, we have compared the experimentally measured elastic curves with the numerically calculated ones, and we can see that the agreement is satisfactory.

CONCLUSIONS

We have studied the deflections of a cantilever beam theoretically, experimentally and numerically. Firstly, we obtained the equations for large deflections, and by analysing this equation students can see that, although they are dealing with a simple physical system, it is described by a differential equation with a nonlinear term. On the other hand, the experiment described in this paper provides students with not only an understanding of geometric nonlinearity but also a better understanding of the basic concepts of mechanics of materials. Important topics, including concentrated and distributed loads, linear elastic materials, modulus of elasticity, large and small deflections, moment-curvature equation, elastic curve, moments of inertia of the beam cross-section or bending moment, are considered in this experiment.

The numerical study using the ANSYS program allows students not only to solve the nonlinear differential equation governing the nonlinear behaviour of the beam, but also to obtain the modulus of elasticity of the material and to compare the experimentally and numerically calculated elastic curves of the beam for different external loads. Finally, we have shown that the geometric nonlinear behaviour of the bending of a cantilever beam may be easily studied with a simple, easy-to-assemble, low-cost experiment, which allows us to experimentally study the deflections of cantilever beams by means of a series of simple measurements, such as lengths and masses.

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