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Numerical Computation of Potential in Unbounded Two-Dimensional Regions using Schwarz-Christoffel Transformation

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Finite difference is a method of choice for educators to demonstrate and compute potential field of two-dimensional geometries, such as integrated circuit planar resistors. It is simple and can be readily programmed by undergraduate students. It is also very accurate and its accuracy can be easily controlled by changing the grid size. Finite difference method (FDM), however, has two serious limitations. One, it can not be easily applied to unbounded regions such as integrated circuit (IC) microstrip lines. And two, the FDM computes potentials at predetermined grid points only. Unlike the finite element method (FEM), it does not generate potential functions that can be used to interpolate for potentials at the points that are not located at the grid, or to use these functions in determining other quantities based upon the computed potential such as the field intensity. This paper describes a hybrid method based upon conformal transformations, including the Schwarz-Christoffel (S-C) transformation (without having to compute the transformation functions) to map the original boundaries, including those at infinity, to a bounded region and only then applies a numerical method based on finite differences. This paper also describes a method that is a combination of the FDM and the FEM to generate the potential functions after the FDM has been applied. The combined method retains the simplicity and accuracy of the FDM. Yet it, like the FEM, provides potential functions that can be used for interpolation as well as post-processing of potential. Testing theses approaches by means of an example for which exact solution is obtainable, the hybrid method and the combination of the FDM-FEM are applied to determine the electrical potential at a specific point in the field of an IC microstrip line. In both cases the results are in agreement with analytically derived results. The approach we have developed is simple, readily applied by undergraduate students, yet accurate and thus of use in professional engineering work.

INTRODUCTION

ONE OF THE most important tools available to educators, students and practicing engineers that can be used to solve electrical potential problems in two dimensions (potential field does not vary with respect to the third dimension) is a method commonly known as Finite Difference Method (FDM) [1, 2]. The FDM is based upon solving Laplace's equation with given boundary conditions by discretizing it, as explained in the following section.

The FDM is very powerful in a sense that it can be applied to a wide range of microelectronic devices of arbitrary geometrical shapes, such as integrated circuit (IC) planar resistors [3, 8]. It is very simplistic, easy to apply and very convenient to program in virtually any programming language and its accuracy can be easily controlled.

The simplicity of the FDM makes it a very effective educational tool in the hands of educators for demonstration of electrical field concepts in microelectronic devices [1]. Students of engineering and physics can easily use it to solve two-dimensional potential field problems for various geometrical configurations with different boundary conditions, such as IC resistors. Because of its accuracy and ease of use, the FDM also remains a powerful tool in the arsenal of design engineers trying to predict electrical, magnetic, and other potential field distributions in their products.

The FDM, however, has two very serious limitations:

1. The FDM is not easily applicable to an unbounded region, such as integrated circuit transmission lines (strip lines, microstrip lines, etc.). The application of the FDM to an unbounded region will result in infinite number of simultaneous equilibrium conditions that obviously are impossible to solve without truncating the region.
2. The FDM only provides numerical values of the potential at the pre-determined grid points. Post-processing of the potential values to determine such quantities as equipotential lines, or electric field intensity, $\vec{E}$, is difficult because the FDM does not generate potential functions [1]. Similarly, the lack of potential functions makes it difficult to interpolate for a potential at a point that is not a grid point. Post-processing of computed potentials is important for educators

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to demonstrate such concepts as flow of flux in an IC microstrip line and flow of current in an IC resistor [3].

One possible way to circumvent the first limitation that may work in some cases is to assume that the potential is zero at a boundary sufficiently far away, thus converting the infinite region into an enclosed finite region with an outer boundary having a boundary condition of zero potential. The problem with this approach however, is that the prediction of a distance at which the potential can be assumed to be zero is merely a guess and can lead to inaccurate results. To assure accuracy, the process has to be repeated until results converge to a satisfactory degree.

In this paper, we describe methods to circumvent both limitations without sacrificing accuracy or relying on guesswork. The methods proposed are simple so can be readily used by undergraduate students of engineering, yet accurate thus can be used by practicing engineers.

The first limitation of the FDM is circumvented by a method that is a hybrid of conformal transformations, including the Schwarz-Christoffel transformation [3, 4] and the Finite Difference Method. In this *hybrid method*, conformal transformation is used to map an unbounded region into a bounded region before the application of the FDM.

The unbounded region in which the device is located (e.g., an IC microstrip line whose potential field is to be determined) is divided into two parts by placing an imaginary closed path (e.g. square) around the device. The resulting two parts are the inside of the square (a bounded region) and the outside of the square (an unbounded region). A set of conformal transformations maps the *outside* of this square (the unbounded part) to the *inside* of another square. This converts an unbounded region into a bounded region.

The FDM is then applied to the interior of both the squares together. This is now equivalent to applying the FDM to the entire original unbounded region in which the device was located.

The second limitation of the FDM is circumvented by the use of finite element method (FEM) in combination with the FDM [10, 11]. The combined FDM-FEM method uses FDM to compute potentials at the pre-determined grid points. Then an approach based on FEM is used to generate *potential functions* that in turn can be used to determine the potential at any other point. The combined method is simple to use compared to applying the FEM directly (which is not usually taught at undergraduate level.) Yet, it is very powerful because just like the FEM, it determines a set of simple mathematical functions that describe the potential field at all points within the geometry.

Examples of the application of the methods are presented. The results obtained are compared with the results acquired from using analytical methods.

THE FINITE DIFFERENCE METHOD

Potential in a two-dimensional geometry that is free of charges is described by the Laplace's equation [1, 13]:

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \tag{1}$$

Figure 1 shows an example of an IC planar resistor, a two-dimensional bounded region [12]. Either the potential is known on the boundaries, or the Neumann boundary condition (Equation 2) exists on some of the boundaries as shown in the figure [5, 8]:

$$\frac{\partial V}{\partial n} = 0 \tag{2}$$

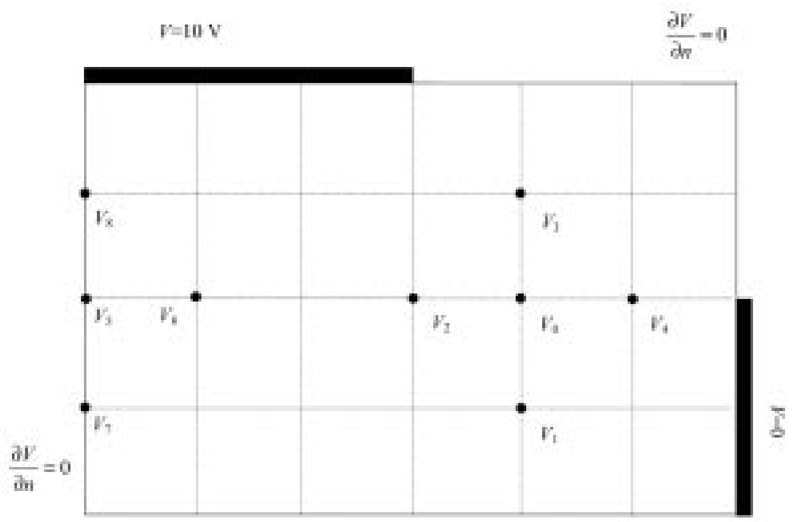

Fig. 1. An integrated circuit planar resistor. Some of the boundaries have known potential while others have Neumann boundary conditions.

The potential at the inside of this bounded region is to be determined.

The region within the boundaries is divided into small squares of equal size creating a grid. As shown in Fig. 1, each grid point has four adjacent grid points. The potential at the center point is given by [1, 2]:

$$V_0 \cong \tfrac{1}{4}(V_1 + V_2 + V_3 + V_4) \tag{3}$$

The Neumann boundary condition is implemented [2] by assuming that the point adjacent to the Neumann boundary, for instance V_6, has a mirror image across the boundary, therefore, V_6 will appear twice when Equation (3) is applied to V_5.

$$V_5 \cong \tfrac{1}{4}(V_7 + V_8 + 2V_6) \tag{4}$$

Equations (3) and (4) are appropriately applied to each point of the grid where the potential is unknown. This results in a number of equations equal to the number of unknown potentials. These equations can be solved by a well-established iterative scheme. First, some estimated or guessed values are assigned to the unknown potentials. These values are then substituted into the right-hand side of these equations. New potential values obtained thereafter are substituted back into the equations to obtain yet newer and better potential values. The iterative process is repeated several times until convergence is reached (difference of potential values computed at all the unknown points after the two most recent iterations are within some predetermined small number, δ).

The Finite Difference method is well suited for digital computers and is easy to program. The accuracy of the method can be increased by *decreasing* the size of the squares (resulting in more grid points) or decreasing δ, or both.

CONFORMAL MAPPING TO BRING INFINITY TO ZERO

Let there be a square in z-plane ($z = x + iy$) as shown in Fig. 2. The square is centered at the origin with corners at (a, a), $(a, -a)$, $(-a, -a)$ and $(-a, a)$. The square divides the entire z-plane into two regions. The *interior* of the square is a *bounded* region while the *exterior* is an *unbounded* region. The exterior of the z-plane square can be mapped to the interior of another identical square in w-plane ($w = u + iv$ or $\sigma e^{i\beta}$) by a complex function [4, 7].

$$z = f(w) \text{ or } w = f^{-1}(z) \tag{5}$$

Equation (5) maps points z_j $(0 \leq j \leq 9)$ to the points w_j $(0 \leq j \leq 9)$, respectively, as shown in Fig. 2. The $z =$ point has mapped to $w = 0.0$. The interior of both the squares now represents the *entire* z-plane, bringing infinity of the z-plane to the origin of the w-plane.

Now, grids can be constructed inside both the squares and the FDM can be applied as demonstrated in the following examples. (It is to be noted that since potential is zero at $z = \infty$, $w = 0.0$ will have a boundary condition of $V = 0.0$.)

The functions f and f^{-1}

The function f, and similarly, f^{-1} are actually not single functions but each is a set of three functions. The first is a Schwarz-Christoffel transformation that maps the z-plane square to a unit

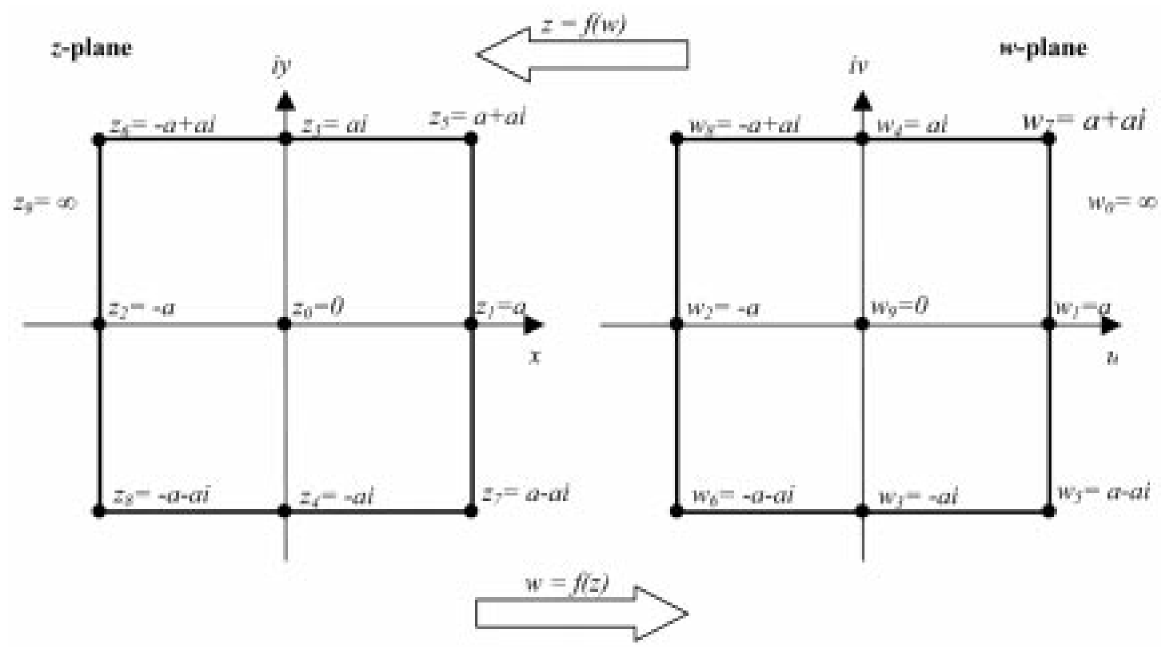

Fig. 2. Mapping of the z-plane square to the w-plane square and vice versa.

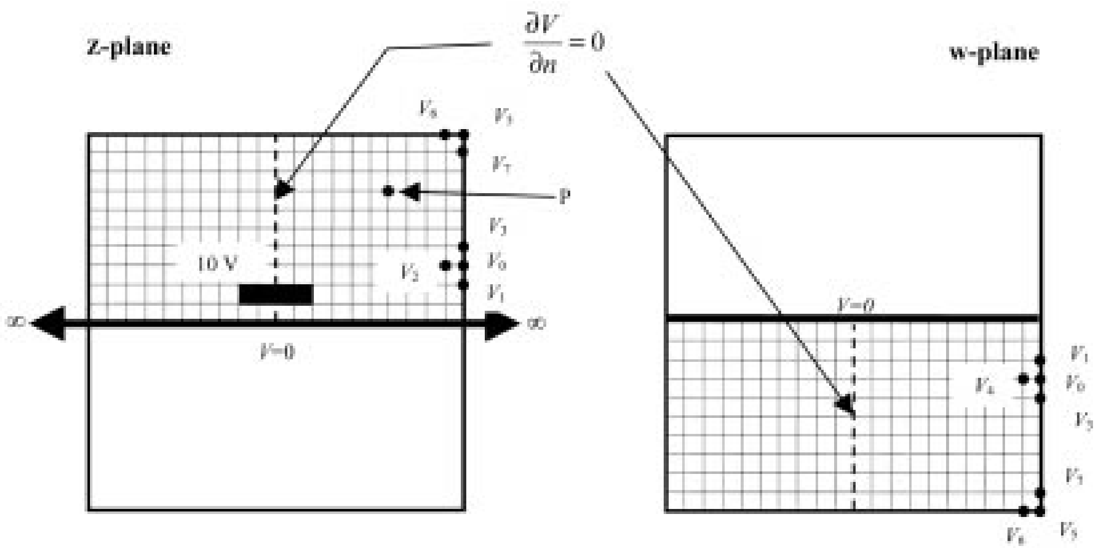

Fig. 3. Cross-section of the integrated circuit transmission (microstrip) line.

circle in an intermediate plane, say, ξ-plane [3–5]. The second is an *inversion function* [6] that maps the ξ-plane circle to another unit circle in, say, ξ-plane. The inversion function inverts the circle and maps the outside of the ξ-plane circle (the unbounded region) to the inside of ξ-plane circle. This converts an unbounded region into a bounded region. The third is also a Schwarz-Christoffel transformation that maps the ξ-plane circle to the w-plane square. The three mapping functions combined together form the function f and its inverse of Equation (5).

The inversion function that maps an outside of ξ-plane circle into the inside of a ξ-plane circle, and vice versa, is straight forward and can be found in any undergraduate level textbook on complex variables [6, 8]. The Schwarz-Christoffel transformation that maps a square onto a circle and vice versa, although well established, is difficult to compute and generally beyond the scope of undergraduate students of engineering and sciences [3, 7, 9].

An interesting aspect of the approach described here is that one does not need to compute these functions. If the points at which the potential is to be determined are located outside the z-plane square, then we must know f^{-1} to locate their images inside the w-plane square to obtain their potential. However, computation of the functions f and f^{-1} can be avoided altogether if such points lie inside the z-plane square. This can be accomplished by judiciously choosing the boundaries of the z-plane square (the value of 'a') such that the square encompasses all such points. Mere existence of such functions that compress an infinite region (exterior of the z-plane square) into a finite region (interior of the w-plane square) bringing infinity of the z-plane to the origin of the w-plane gives us the necessary theoretical justification to solve the potential problem numerically inside both the squares, without having to compute these functions.

APPLICATION OF THE METHOD

The potential field of an integrated circuit microstrip line over a ground plane embedded in a uniform medium is to be determined [7]. The microstrip line consists of an infinite ground plane and a parallel strip having a thickness of 0.001 m, width of 0.004 m, located 0.001 cm above the ground plane and having a potential of 10 V. Since we know the analytical solution of this example, it provides a good test for our approach.

The microstrip line is placed in the z-plane, parallel to the x-axis with the ground plane at the $x = 0$ axis, as shown in Fig. 3. We will compute the potential at a test point P (0.006, 0.007 m). The z-plane square is drawn with its center at the origin. The 'a' is chosen to be 0.02 m so that P is enclosed within the square. A corresponding square is drawn in the w-plane. Note that the ground plane ($x = 0$) *outside* the z-plane square has mapped on to the real axis *inside* the w-plane

Table 1. Potential of point P using various methods for comparison

	Potential at point P	Percentage error
Potential computed using hybrid of Finite Difference method and S-C Transformation	1.6112 V	1.468%
Potential computed by assuming zero potential at the boundary of square in the z-plane	1.0115 V	38.14%
Potential computed analytically [3, 7]	1.6352 V	–

square. Grids of squares of size 0.001 m are created inside of both the squares. Equations (3) and (4) are appropriately applied to each of the grid points.

Because of the horizontal symmetry, a potential problem in only quarter regions of both the squares needs to be solved. The real axes of both the squares have boundary condition of $V = 0.0\,\mathrm{V}$ whereas Neumann boundary condition (Equation 2) exists at the vertical axes.

Special consideration is given when Equations (3) and (4) are applied to the grid points located on the boundary of the z-plane (hence w-plane) squares (Fig. 3).

For instance, when Equation (3) is applied to the point V_0, the two of the four adjacent points, V_1 and V_3, are located on the boundary of the squares, whereas V_2 is located inside of the z-plane square and V_4 is located inside the w-plane square. Similarly, the points located at the corner have only two adjacent points. For insistence, the adjacent points to V_5 are V_6 and V_7. Therefore, Equation (3) is modified to:

$$V_5 = \frac{2V_6 + 2V_7}{4} = \frac{V_6 + V_7}{2} \qquad (6)$$

The $\delta = 0.001$ was used to compute the potential of point P.

For comparison, the potential of the same point was determined using finite difference method using the same grid size and value of δ. However, instead of using the conformal mapping approach, the potential was assumed to be zero at the boundaries of the square.

The results are compared with the potential derived analytically and are presented in Table 1 [3, 7].

THE COMBINATION OF FDM AND FEM

In the above example the point P is a grid point in the z-plane square. If the point is not located on the grid, then one option is to reconfigure the grid so that the point falls on the grid, a potentially laborious work. Alternately, we can determine a set of *potential functions* that can be used to interpolate potential at a*ll* the points *after* the potential has been computed at the grid points.

To generate potential functions, each square of the grid used in the finite difference method is divided into two right angle triangles called elements, as shown in Fig. 4.

Figure 5 shows one of the above triangular elements placed in a co-ordinate system.

A linear polynomial (Equation 7) can be used to describe the potential function for each element [2]:

$$V(x, y) = a_0 + a_1 x + a_2 y \ (V) \qquad (7)$$

The constants a_0 , a_1 and a_2 are found by expressing Equation (7) in terms of now known potentials at the nodes of the element. The unknown constants of the Equation (7) for the element shown in Fig. 5 can be computed by the relationships:

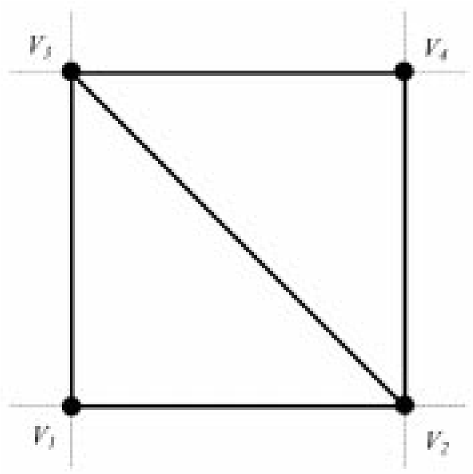

Fig. 4. A square grid is split into two triangular elements.

$$V_1 = a_0 + a_1 x_1 + a_2 y_1 \qquad (8a)$$
$$V_2 = a_0 + a_1 x_2 + a_2 y_2 \qquad (8b)$$
$$V_3 = a_0 + a_1 x_3 + a_2 y_3 \qquad (8c)$$

The Equations (8a) to (8c) are solved for unknown constants. The solution is expressed in Equations (9a) to (9d):

$$a_0 = \frac{1}{2A_e}[V_1(x_2y_3 - x_3y_2) + V_2(x_3y_1 - x_1y_3) + V_3(x_1y_2 - x_2y_1)] \ (\mathrm{V}) \qquad (9a)$$

$$a_1 = \frac{1}{2A_e}[V_1(y_2 - y_3) + V_2(y_3 - y_1) + V_3(y_1 - y_2)] \ (\mathrm{V/m}) \qquad (9b)$$

$$a_2 = \frac{1}{2A_e}[V_1(x_3 - x_2) + V_2(x_1 - x_3) + V_3(x_2 - x_1)] \ (\mathrm{V/m}) \qquad (9c)$$

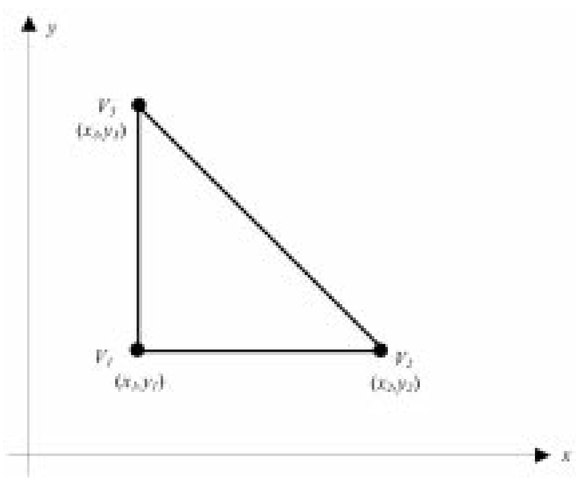

Fig. 5. An element in x-y coordinates.

where A_e is the area of the triangular element:

$$A_e = \tfrac{1}{2}[(x_2y_3 - x_3y_2) + (x_3y_1 - x_1y_3) + (x_1y_2 - x_2y_1)]\ (\mathrm{m}^2) \tag{9d}$$

Equations (7) to (9) can be applied to each of the triangular elements of the two-dimensional geometry resulting in a potential function for each element. These potential functions can now be used to interpolate potential values as well as to compute quantities such as electric field intensity $\vec{E}$ within their corresponding elements.

Since we are using the Finite Element Method after the FDM, computational effort involved is much smaller compared with the applying the FEM directly [10, 11]. The combined approach is also considerably simpler and can be readily used by the undergraduate students of engineering and science.

APPLICATION OF THE COMBINED METHOD

We applied the combined FDM-FEM method to the triangular element adjacent to point P in Fig. 3 (the point P is located at the lower left, right angled corner of the element as shown in Fig. 3.) The application of Equations (7) to (9) to the element resulted in the following potential function (x and y are in meters):

$$V(x, y) = 54 - 4257x - 1764y\ (\mathrm{V}) \tag{10}$$

Now we can use this function to find the potential at any other point inside that element. We can also use it to find quantities such as electric field intensity, $\vec{E}$, which is given by:

$$\vec{E} = -\frac{\partial V}{\partial x}e_x - \frac{\partial V}{\partial y}e_y \tag{11}$$

The application of Equation (11) to the potential function of Equation 10 resulted in the following expression of electric field intensity at the point P.

$$\vec{E} = 4257e_x + 1764e_x\ (\mathrm{V/m}) \tag{12}$$

The analytic computation of the electric field intensity results in the following value:

$$\vec{E} = 4262e_x + 1785e_x\ (\mathrm{V/m}) \tag{13}$$

As can be seen, the analytic value is in agreement with the computed value.

FUTURE WORK

One possible area of future work can be the development of algorithms that combine conformal mapping functions directly with the finite element method. First, conformal mapping functions are used to convert an unbounded region into a bounded region, then finite element analysis is used directly to solve for the potential problem in the bounded region [10].

Of special interest to students is the use of physical analogs. Thus one can fashion a current flow analog by using a combination of a low conductivity medium simulating 'space' with a high conductivity medium simulating the conducting elements or electrodes. 'Infinity' would be simulated by bringing the outer boundary of the analog two-dimensional field in contact with another analog that simulates the field beyond the region under investigation. This outer region can be represented by another conducting surface with a solid (or piecewise solid) connection between the identical points on the outer boundaries of both *dishes*, with a sink at the center of this second plane.

CONCLUSION

We have described a technique that is *hybrid* of the conformal mapping and finite difference method. It relies on conformal mapping to convert an unbounded region into a bounded region. Then the finite difference method is used to solve for the potential.

An interesting aspect of the hybrid method is that although conformal mapping provides the theoretical rational, one does not need to compute the mapping function. The z-plane square can be judiciously chosen to encompass all the points at which potential needs to be computed thus avoiding the need of the mapping function.

Through an example, we have demonstrated that the hybrid method is more accurate than using the FDM alone. Yet it is simple enough that undergraduate students of engineering and physical sciences can easily program and apply it.

We have also proposed a method to combine the finite difference method with the finite element method. We have demonstrated the combined method through a practical example.

The combined method retains the simplicity and ease of use of the FDM. Yet, it generates potential functions that can be used for interpolation and post-processing. The combined method can be easily used by the undergraduate students of engineering and physical sciences who usually do not have enough background to use the finite element method directly.

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