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Procedure to calculate deflections of curved beams*

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In the study presented here, the problem of calculating deflections of curved beams is addressed. The curved beams are subjected to both bending and torsion at the same time. The Castigliano theorem, taught in many standard courses in Strength of Materials, Mechanics of Solids, and Mechanics of Materials, is used to determine the beam deflections. Using the methodology presented here, beam deflections that cannot be found in handbooks or textbooks can be calculated without too much effort. The Castigliano theorem and a numerical integration algorithm from the MATLAB package have been used. The examples investigated in this paper deal with elliptically curved beams. The beams are either statically determinate or statically indeterminate. Limiting cases of the elliptical beam are bending of straight beams and bending and torsion of a circular beam. Beam deflections obtained in the limiting cases are compared with handbook formulae.

AUTHOR'S QUESTIONNAIRE

1. Solution methods discussed in this paper are of interest for mechanical and civil engineering education where bending and torsion of straight and curved beams are taught.
2. The Castigliano theorem is used to solve one class of problems that cannot easily be solved using other methods, including the finite element method.
3. Bending and torsion of curved beams are investigated. It is demonstrated that these problems can be solved without too much effort.
4. Commonly used beam bending formulae are obtained as limiting cases.
5. Using, for example, the MATLAB package, the student may practice numerical calculations.
6. Two problems, one statically determinate and one statically indeterminate, are analysed and discussed.

INTRODUCTION

MANY BASIC COURSES in solid mechanics and/or strength of materials given for mechanical and civil engineering students often include the concepts work and elastic strain energy. Using these concepts, methods for analysing the behaviour of elastic structures have been developed. In this paper the well known theorem by Castigliano (Castigliano's second theorem) will be used in association with a numerical integration algorithm to solve one class of problems that cannot easily be solved by analytical methods or by the finite element method. It is demonstrated how the Castigliano theorem can be used to calculate deflections of curved beams, both statically determinate and statically indeterminate.

The curved beams investigated in this paper will have the form of either a quarter of an ellipse or half an ellipse. The half-axes of the ellipse will be denoted a and b. The load acts normally to the plane of the curved beam. In the first example, the problem is statically determinate. The beam, curved to the form of a quarter of an ellipse, is clamped at one end and free at the other. In the second example, a half-elliptical beam is clamped at both ends, thus giving a statically indeterminate problem.

The quarter-elliptical beam is clamped at one end and loaded by a force P at the free end. The force acts perpendicularly to the plane of the curved beam, see Fig. 1. In the limits, when one of the half-axes of the ellipse (a or b) tends to zero, the quarter-elliptic beam tends to a straight cantilever beam loaded by the force P at the free end. Studying bending of beams, this is a standard case found in any textbook in solid mechanics or strength of materials. The deflection δ of the free end of the beam is (linear elastic material is assumed) [1]:

$$\delta = \frac{PL^3}{3EI} \tag{1}$$

where L is the length of the beam (i.e. the length of the ellipse's half-axis not tending to zero) and EI is the bending stiffness of the straight beam. This case will be obtained as a limiting case in the calculations presented below.

When the two half-axes of the elliptical beam are equal (i.e. $a = b = R$) the form of the curved beam will be a quarter of a circle. The deflection δ of the free end of the quarter-circular beam can be found in, for example, the handbook *Roark's formulas for stress and strain* [2]. It becomes:

$$\delta = \frac{\pi}{4}\frac{PR^3}{EI} + \left(\frac{3\pi}{4} - 2\right)\frac{PR^3}{GK_t} \tag{2a}$$

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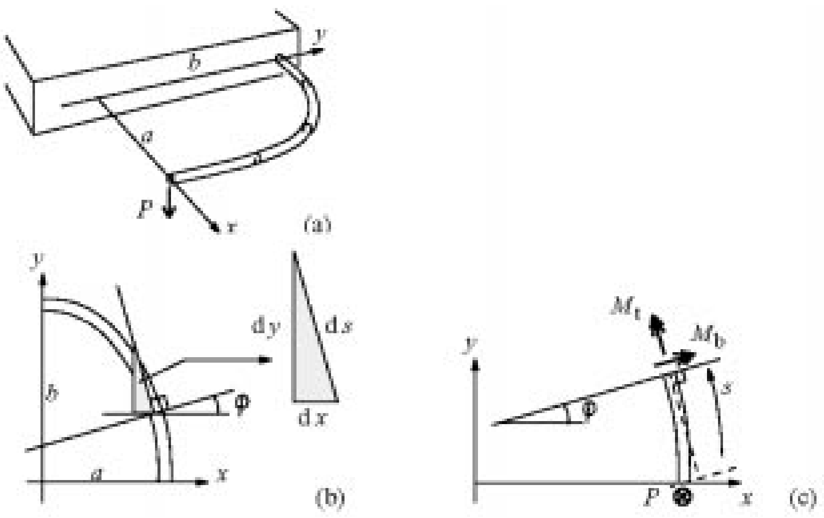


Fig. 1. (a) Curved cantilever beam (uniform cross section) curved to the form of a quarter of an ellipse. (b) Definition of beam geometry, and (c) cross sectional moments: M_b bending moment and M_t twisting moment (torque). Influence of the shear force on beam deflection is neglected (shear force not shown in the figure).

where R is the radius of curvature of the beam. It is assumed that R is much larger than a diagonal measure of the beam cross section, i.e. $R \gg I^{1/4}$, where I is the second moment of the beam cross-sectional area. Further, EI is the bending stiffness (bending rigidity), and GK_t is the torsional rigidity of the beam. Also, E is the modulus of elasticity, $G = E/2(1+\nu)$ is the shear modulus, and ν is the Poisson ratio of the beam material. Also this case, i.e. Equation (2a), will be obtained as a limiting case in the calculations presented below.

Often a beam with a circular cross section, diameter d, is examined. The second moment I of the beam cross-sectional area then is $I = \pi d^4/64$ and the factor K_t in the torsional rigidity of the beam cross-section is $K_t = \pi d^4/32$, and, using $\nu = 0.3$, one obtains, in agreement with [2]:

$$\delta = \frac{PR^3}{EI}\left\{\frac{\pi}{4} + (1+\nu)\left(\frac{3\pi}{4} - 2\right)\right\}$$
$$= 1.2485\frac{PR^3}{EI} \quad \text{where} \quad I = \frac{\pi d^4}{64} \tag{2b}$$

The half-elliptical beam is clamped at the two ends and loaded by a force P perpendicularly to the plane of the curved beam. The force P is applied at the centre of the beam (i.e. in the plane of symmetry), see Fig. 5. This problem is statically indeterminate. In this problem the limiting case when the half-axis b (see Fig. 5) tends to zero will become a double cantilever beam carrying the load P at its free end (a double cantilever beam in the sense that the load P is carried by two straight cantilever beams parallel to each other). The deflection δ at load P then becomes:

$$\delta = \frac{1}{2}\frac{Pa^3}{3EI} \tag{3a}$$

where a is the length of the two cantilever beams and the factor 1:2 is there because the load P is carried by two parallel beams.

In the other extreme case, when the half-axis a tends to zero (see Fig. 5), a straight beam of length $2b$ that is clamped at the two ends is obtained. The force P at the middle of the beam then causes the beam centre to deflect the distance [1]:

$$\delta = \frac{P(2b)^3}{64 \cdot 3EI} = \frac{Pb^3}{24EI} \tag{3b}$$

The third limiting case, $a = b$, gives a beam with a circular curvature; the beam takes the form of half a circle. This case is less frequent in the literature; only [3] has been found. One has:

$$\delta = \frac{PR^3}{2EI}\left(\frac{\pi}{4} - \frac{1}{\pi}\right) + \frac{PR^3}{2GK_t}\left(\frac{3\pi}{4} - \frac{1}{\pi} - 2\right) \tag{3c}$$

For a beam with circular cross section, diameter d, the expression (3c) simplifies to ($\nu = 0.3$):

$$\delta = 0.2582\frac{PR^3}{EI} \quad \text{where} \quad I = \frac{\pi d^4}{64} \tag{3d}$$

Solutions to the problems solved and discussed in this paper have not been found in the literature. The solutions are interesting from an educational point of view, because the problems solved tend to three known solutions for special cases of the minor and major axes of the ellipse.

STATICALLY DETERMINATE PROBLEM

In this section a cantilever beam, curved to the form of a quarter of an ellipse, will be investigated. The beam is clamped at one end and loaded with a force P at the free end, see Fig. 1(a). The force P acts perpendicularly to the plane of the ellipse. Let a be the length (in the x direction) of one half-axis of the ellipse and b the length of the other half-axis (in the y direction). In the extreme case $a = 0$, the beam will then be a straight cantilever beam of length b, and in the case $b = 0$, the beam will be a straight cantilever beam of length a. In the third case, $a = b$, the beam will take the form of a quarter of a circle. Sometimes, but not very often, the out-of-plane bending of such a beam may be treated in textbooks, see for example [4] and [5].

The bending stiffness of the curved beam is EI and the torsional rigidity is GK_t (uniform along the beam). The material is linear elastic; E is the modulus of elasticity (Young's modulus) and G is the shear modulus. The second moment of the cross-sectional area is denoted I, and K_t is the cross-sectional factor of the torsional rigidity.

The deflection δ of the beam end (at the point of application of the force P and in the direction of the force) will be determined. The force P is normal to the xy plane. Also, for axis b the notation $b = \beta a$ is introduced. Here β could be larger than or smaller than 1.

Especially, the cases $\beta = 0$, $\beta = 1$ and $\beta \gg 1$ will be investigated and the results will be compared with known results for straight cantilever beams of length a and b, respectively, and for the quarter-circular cantilever beam (with $a = b = R$, and R is the radius of the circularly curved beam). In some cases it is also assumed (for simplicity) that the beam has a circular cross-section with diameter d, where $d \ll a$ and/or b, implying that beam theory for straight beams can be applied.

Solution

First, the equation of the ellipse is examined. The equation reads:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{4}$$

The beam studied here is located in the first quadrant of the coordinate system, so here $0 \le x \le a$ and $0 \le y \le b$, where a and b are the half-axes of the ellipse, see Fig. 1. Solve (4) for y. It gives:

$$y = b\sqrt{1 - \frac{x^2}{a^2}} = \beta a\sqrt{1 - \frac{x^2}{a^2}} \tag{5}$$

where $b = \beta a$ has been introduced. Differentiation of Equation (5) gives:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-\beta x/a}{\sqrt{1 - x^2/a^2}} = -\frac{b^2 x}{a^2 y}\left\{= \tan\left(\frac{\pi}{2} + \varphi\right)\right\} \tag{6a, b, c}$$

One notices that $\mathrm{d}y$ and $\mathrm{d}x$ have different signs because the expression (6a) is always negative. Here $\mathrm{d}x$ is negative. It is also noticed that the length $\mathrm{d}s$ of a beam element is:

$$\mathrm{d}s = \sqrt{(\mathrm{d}x)^2 + (\mathrm{d}y)^2} = -\mathrm{d}x\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} \tag{7}$$

The negative root has been selected because $\mathrm{d}x$ is negative while the length $\mathrm{d}s$ is positive. Also $\cos\varphi$ and $\sin\varphi$ will be needed. One obtains:

$$\sin\varphi = -\frac{\mathrm{d}x}{\mathrm{d}s} = \frac{+1}{\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}} \quad \text{and}$$

$$\cos\varphi = -\frac{\mathrm{d}y}{\mathrm{d}s} = \frac{-\mathrm{d}y/\mathrm{d}x}{\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}} \tag{8a, b}$$

The angle φ varies between 0 and $\pi/2$, which implies that both $\cos\varphi$ and $\sin\varphi$ are positive.

Next, study the beam cross-section situated at angle φ. At this cross-section the bending moment $M_{\text{bending}} = M_b$ and the torsional (twisting) moment $M_{\text{torsion}} = M_t$ are acting, see Fig. 1(c). The shear force has been omitted in the figure; its influence on the beam deflection will be neglected in the calculations performed here. The equilibrium equations will be established. Here the equations of moment equilibrium are used. At the cross-section at angle φ, the directions x and y are selected as directions for the moment equilibrium equations. Then the shear force will not appear in the equations. One obtains:

$$\begin{aligned} M_b\cos\varphi - M_t\sin\varphi + Py &= 0 \\ M_b\sin\varphi + M_t\cos\varphi + P(a - x) &= 0 \end{aligned} \tag{9a, b}$$

Solving for M_b and M_t gives:

$$\begin{aligned} M_b &= -Py\cos\varphi - P(a - x)\sin\varphi \\ M_t &= +Py\sin\varphi - P(a - x)\cos\varphi \end{aligned} \tag{10a, b}$$

The elastic strain energy stored in the beam can now be determined. One has:

$$U = \frac{1}{2EI}\int_0^L M_b^2\,\mathrm{d}s + \frac{1}{2GK_t}\int_0^L M_t^2\,\mathrm{d}s \tag{11}$$

where L is the length of the beam (the length L need not be calculated, because the integration will be performed over the variable x and not over s). The contribution of the shear force to the strain energy U has been neglected. Using the Castigliano theorem (the second theorem), the deflection δ of the beam end at the load P can be calculated. One obtains:

$$\delta = \frac{\partial U}{\partial P} = \frac{1}{2EI}\int_0^L 2M_b\frac{\partial M_b}{\partial P}\,\mathrm{d}s + \frac{1}{2GK_t}\int_0^L 2M_t\frac{\partial M_t}{\partial P}\,\mathrm{d}s \tag{12}$$

Enter M_b, $\partial M_\mathrm{b}/\partial P$, M_t, and $\partial M_\mathrm{t}/\partial P$ from (10) into (12). It gives:

$$\delta = \frac{P}{EI}\int_0^L (-y\cos\varphi - (a-x)\sin\varphi)^2 \mathrm{d}s$$

$$+\frac{P}{GK_\mathrm{t}}\int_0^L (y\sin\varphi - (a-x)\cos\varphi)^2 \mathrm{d}s \qquad (13)$$

Next, enter into (13) the expressions of $\cos\varphi$, $\sin\varphi$, y, $\mathrm{d}y$, and $\mathrm{d}s$ as given in Equations (6) to (8) as function of the variable x. The integration over $\mathrm{d}s$ from 0 to L then becomes an integration over $\mathrm{d}x$ from a to 0. Change the order of the integration limits (thus, integrate from 0 to a) and change the sign of the integrand. Also, introduce $b=\beta a$ and remove a^3 from the integrals. One obtains, with x/a as a new dimensionless integration variable, giving integration limits 0 and 1:

$$\delta = \frac{Pa^3}{EI}I_1(\beta) + \frac{Pa^3}{GK_\mathrm{t}}I_2(\beta) \qquad (14\mathrm{a})$$

where the integrals I_1 and I_2 are functions of the parameter β $(=b/a)$ only. One obtains:

$$I_1(\beta) = \int_0^1 \left(\beta\sqrt{1-\frac{x^2}{a^2}}\frac{\mathrm{d}y/\mathrm{d}x}{\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}} - \left(1-\frac{x}{a}\right)\frac{1}{\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}}\right)^2 \sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}\left(\frac{x}{a}\right) \qquad (14\mathrm{b})$$

and

$$I_2(\beta) = \int_0^1 \left(\beta\sqrt{1-\frac{x^2}{a^2}}\frac{1}{\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}} + \left(1-\frac{x}{a}\right)\frac{\mathrm{d}y/\mathrm{d}x}{\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}}\right)^2 \sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}\left(\frac{x}{a}\right) \qquad (14\mathrm{c})$$

An expression giving the deflection of the elliptically curved cantilever beam has now been found. Each one of the two terms in Equation (14a) will be investigated, i.e., it will be investigate how bending and torsion, respectively, contribute to the deflection δ.

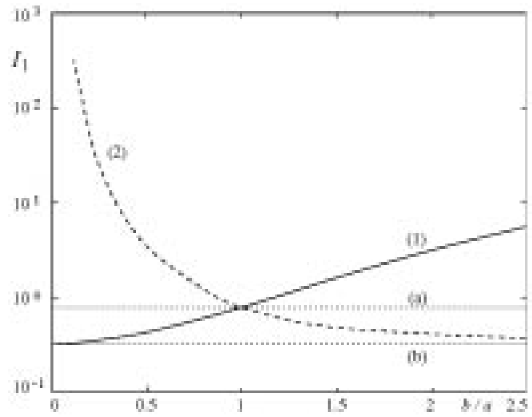


Fig. 2. Curve (1): Integral $I_1=\delta/(Pa^3/EI)$ as function of parameter $\beta=b/a$. It is seen that for $\beta=0$ one obtains $I_1=1/3$ (reference line (b) at 1/3), which is the deflection of a straight cantilever beam of length a, see Equation (1). For $\beta=1$ one obtains $I_1=\pi/4$ (reference line (a), cf. the bending contribution to Equation (2a)). Curve (2) shows I_1/β^3. One notices that for large values of β one obtains $I_1/\beta^3=1/3$, i.e., the same result as for bending of a straight cantilever beam of length $b=\beta a$. The two curves intersect at $\beta=1$, as they should.

Influence of bending

First, investigate the integral I_1. After simplification, one obtains:

$$I_1(\beta) = \int_0^1 \left(\beta\sqrt{1-\frac{x^2}{a^2}}\frac{\mathrm{d}y}{\mathrm{d}x} - \left(1-\frac{x}{a}\right)\right)^2 \times \frac{1}{\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}}\,\mathrm{d}\left(\frac{x}{a}\right) \qquad (15\mathrm{a})$$

where $\mathrm{d}y/\mathrm{d}x$ is given in Equation (6a).

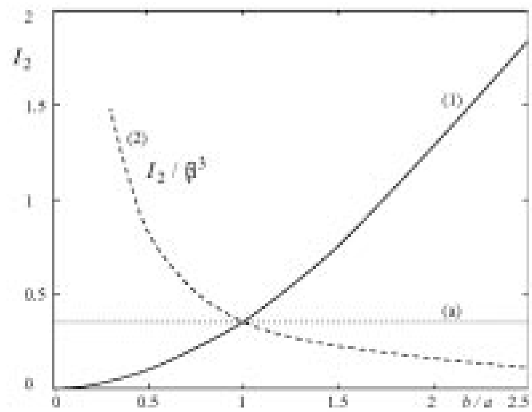


Fig. 3. Curve (1): Integral $I_2=\delta/(Pa^3/GK_\mathrm{t})$ as function of parameter $\beta=b/a$. One notices that $I_2=0$ for $\beta=0$, i.e., the torsion does not contribute to the deflection when $\beta=0$. This was expected, because $\beta=0$ gives a straight cantilever beam loaded only in bending. When $\beta=1$ one obtains $I_2=3\pi/4-2=0.3562$ (reference line (a), cf. the torsional contribution to Equation (2a)). This agrees with what can be found in some handbooks. Curve (2) shows I_2/β^3. One notices that for large values of β the factor I_2/β^3 tends to zero, i.e., the torsion does not contribute to the deflection when β is very large. Also this was expected because a very large value of β gives a straight cantilever beam loaded only in bending. The two curves intersect at $\beta=1$, as they should.

Examine the three cases $\beta = 0$, $\beta = 1$ and $\beta \gg 1$. The case $\beta = 0$ gives $dy/dx = 0$, and one obtains:

$$I_1(\beta = 0) = \int_0^1 \left(1 - \frac{x}{a}\right)^2 d\left(\frac{x}{a}\right) = \frac{1}{3} \qquad (15b)$$

This result, together with Equation (14a), agrees with the deflection obtained when a straight cantilever beam of length a is loaded with a force P at the free end. In this case one has $\delta = Pa^3/3EI$, as expressed in Equation (1). Note that here only the term in Equation (14) describing the bending has been investigated. As will be shown below, see Fig. 3, the torsion will not contribute to the deflection when $\beta = 0$.

The case $\beta = 1$ gives a beam with a circular curvature. This case can be found in some textbooks and handbooks. Entering dy/dx from (6a) (using $\beta = 1$) into (15a) gives:

$$I_1(\beta = 1) = \int_0^1 \left(\sqrt{1 - \frac{x^2}{a^2}} \frac{-x/a}{\sqrt{1 - x^2/a^2}} - \left(1 - \frac{x}{a}\right) \right)^2 \times \sqrt{1 - \left(\frac{x}{a}\right)^2} d\left(\frac{x}{a}\right) = \int_0^1 \sqrt{1 - \left(\frac{x}{a}\right)^2} d\left(\frac{x}{a}\right) \qquad (15c)$$

Let $x/a = \xi$. It gives:

$$I_1(\beta = 1) = \int_0^1 \sqrt{1 - \xi^2}\, d\xi = \frac{1}{2}\left[\xi\sqrt{1 - \xi^2} + \arcsin \xi\right]_0^1 = \frac{1}{2}\arcsin 1 = \frac{\pi}{4} \qquad (15d)$$

This result can be compared with that given in some textbooks and recapitulated in Equation (2a), namely $\delta = \pi PR^3/4EI$, see the first term on the right hand side of expression (2a), where the part of the deflection depending on bending is given. Entering Equation (15d) into (14a) gives, as it should, the deflection $\pi Pa^3/4EI$ for the term representing the bending.

In the case $\beta \gg 1$ we start to multiply the numerator and the denominator in Equation (14a) with β^3. Doing this, the expression $P(\beta a)^3/EI$ ($= Pb^3/EI$) is obtained as a factor in front of the integral at the same time as the integral I_1 is divided by β^3. Entering dy/dx from (6a), one obtains:

$$\frac{1}{\beta^3} I_1(\beta) = \int_0^1 \left(\sqrt{1 - \xi^2} \frac{-\beta\xi}{\sqrt{1 - \xi^2}} - \frac{1}{\beta}(1 - \xi) \right)^2 \times \frac{1}{\sqrt{1 + \left(\frac{-\beta\xi}{\sqrt{1 - \xi^2}}\right)^2}} d\xi = \int_0^1 \left(\xi + \frac{1}{\beta^2}(1 - \xi) \right)^2 \times \frac{\beta^2\sqrt{1 - \xi^2}}{\beta^2\sqrt{\frac{1}{\beta^2}(1 - \xi^2) + \xi^2}} d\xi \qquad (15e)$$

As $\beta \gg 1$ the expression (15e) can be approximated. Omitting small terms one obtains:

$$\frac{1}{\beta^3} I_1(\beta \gg 1) = \int_0^1 \xi^2 \frac{\sqrt{1 - \xi^2}}{\sqrt{\xi^2}} d\xi = \int_0^1 \xi\sqrt{1 - \xi^2}\, d\xi = [-\tfrac{1}{3}(1 - \xi^2)^{3/2}]_0^1 = \tfrac{1}{3} \qquad (15f)$$

Together with (14a) this result is in agreement with the deflection of a straight cantilever beam of length $\beta a = b$. Also here the term expressing the influence of torsion of the beam tends to zero when $\beta \gg 1$, see Curve (2) in Fig. 3.

For an arbitrary value of β the integral I_1 is solved numerically. This is a suitable exercise for programming in MATLAB and the calculated results can be checked versus the three limiting cases $\beta = 0$, $\beta = 1$, and $\beta \gg 1$. For the bending part of the solution in Equation (14a), calculated results are presented in Fig. 2. It is seen in Fig. 2 that the three limiting cases for the three values of β are obtained.

Influence of torsion

The integral I_2 in (14) will now be investigated. Some simplifications of Equation (14c) gives:

$$I_2(\beta) = \int_0^1 \left(\beta\sqrt{1 - \frac{x^2}{a^2}} + \left(1 - \frac{x}{a}\right)\frac{dy}{dx} \right)^2 \times \frac{1}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}} d\left(\frac{x}{a}\right) \qquad (16)$$

Table 1. The factor $k(\beta)$ in Equation (18) for some values of β ($= b/a$) and for a circular beam cross-section

β	0.0	0.5	1.0	1.5	2.0	3.0	5.0	10.0	100.0
$k(\beta)$	0.333	0.543	1.249	2.618	4.859	12.86	49.60	353.9	$3.337 \cdot 10^5$

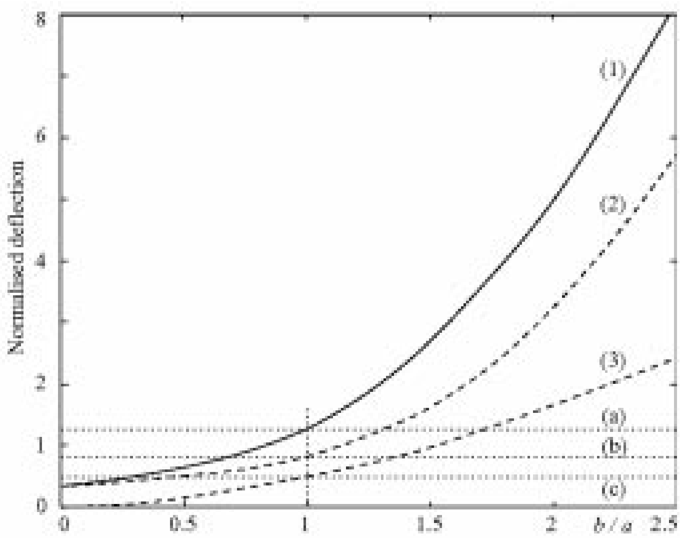


Fig. 4. Curve (1): Normalised deflection $\delta/(Pa^3/EI) = k(\beta)$ of beam end as function of the parameter $\beta = b/a$. The contribution $I_1(\beta)$ to $k(\beta)$ due to bending is given by Curve (2) (the same curve as Curve (1) in Fig. 2, but now the scale on the y-axis is linear). The contribution $1.3I_2(\beta)$ to $k(\beta)$ due to torsion is given by Curve (3) (the same curve as in Fig. 3 but now multiplied by the factor 1.3; i.e., a circular beam cross section has been assumed). The reference lines are situated at the levels (a) 1.2485, cf. Equation (2b), (b) $\pi/4$, and (c) $1.3(3\pi/4 - 2)$. These levels can be checked using handbook solutions.

Numerical integration gives I_2 as presented in Fig. 3, Curve (1). Also, I_2/β^3 has been plotted in Fig. 3, see Curve (2). Curve (2) indicates that I_2/β^3 tends to zero when β becomes large, implying that for $\beta \gg 1$ (a tends to zero) only bending contributes to the beam deflection, see also the discussion of Table 1.

Circular cross-section

For a beam with a circular cross-section one has, using $\nu = 0.3$:

$$GK_t = \frac{E}{2(1+\nu)} 2I = \frac{1}{1.3} EI \tag{17}$$

Enter Equation (17) into the expression (14a). It gives:

$$\delta = \frac{Pa^3}{EI} (I_1(\beta) + 1.3I_2(\beta)) = k(\beta) \frac{Pa^3}{EI} \tag{18}$$

where the factor $k(\beta)$ is given in Fig. 4. In Table 1 the factor $k(\beta)$ is given for some values of β.

It is noted in Table 1 that for $\beta = 0$ and for large values of β the deflections of straight cantilever beams are obtained. When $\beta = 0$ (implying that half-axis $b = 0$) the factor multiplying Pa^3/EI is 1/3 (0.333 in the table), as is should. If $k(\beta)$ is divided by β^3 the factor for a straight cantilever beam with length $b = \beta a$ is obtained, i.e. $\delta/(P(\beta a)^3/EI) = 1/3$. This value is found in the table at $\beta = 100$. The factor becomes $3.337 \cdot 10^5/100^3 = 0.3337 \approx 1/3$, as it should.

A comment on the integration will also be given. The upper integration limit ξ $(= x/a) = 1$ will give division by zero in dy/dx. Therefore integration was performed to $\xi = 0.99999$ only. As the elastic strain energy stored in the beam is finite, the energy stored in the remaining art of the beam can be neglected.

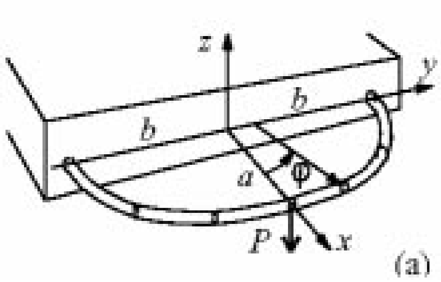
STATICALLY INDETERMINATE PROBLEM

A curved beam (bending stiffness EI and torsional rigidity GK_t) has the form of half an ellipse. The ellipse has the radii (half-axes) a and b. The total length of the beam thus is approximately $L = \pi\sqrt{(a^2 + b^2)/2}$. The ends (the boundaries) of the beam are clamped. The beam is loaded by a force P acting perpendicularly to the plane of the curved beam, see Fig. 5. The structure is symmetric, and the xz-plane is the plane of symmetry. The deflection δ of the beam at the load P will be determined.

Solution

The equation of the ellipse is given in Equation (4). Here the symmetry with respect to the xz-plane will be used in all calculations, so that $0 \le x \le a$ and $0 \le y \le b$. Thus, from now on only half the beam, with appropriate boundary conditions at $\varphi = 0$, will be studied. The quantities y, dy/dx, ds, $\sin\varphi$, and $\cos\varphi$ all have been expressed as functions of x and dx in Equations (5) to (8).

At the load P a sectional bending moment $M_b(\varphi = 0) = M$ is applied, see Fig. 5(b). Due to symmetry, there will not be any torsional moment in the beam at the force P. The shear force in the beam is $Q = P/2$. As only half the beam is investigated, we now have a beam with the form of a quarter of an ellipse. The quarter-elliptical beam is clamped at one end and free to translate in the z direction and free to rotate with respect to the y direction at the other end. Due to symmetry the slope (rotation with respect to the x direction) is zero at this end. The sectional moment M (at $\varphi = 0$) will thus be determined so that it gives the slope zero of the beam end. At an arbitrary cross-section of the beam, determined by the coordinate φ $(0 \le \varphi \le \pi/2)$, the cross-sectional reactions



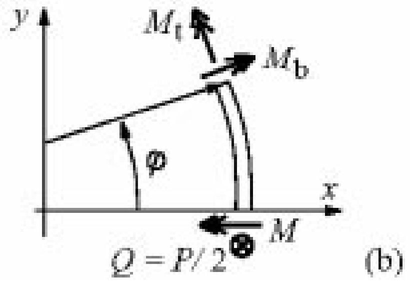


Fig. 5. (a) A clamped/clamped beam curved to the form of half an ellipse. (b) Sectional moments M_b (bending) and M_t (torsion) as function of angle φ. At the plane of symmetry ($\varphi = 0$) only the bending moment $M_b(\varphi = 0) = M$ is acting (no torsion due to symmetry). The load P is applied in the plane of symmetry giving shear force $Q = P/2$ on each half of the beam (shear force at φ not shown in the figure).

(bending moment and torsional moment) M_b and M_t and a shear force are acting. The shear force is not shown in Fig. 5(b), and it need not be determined because its influence on the deflection will be neglected.

The cross-sectional moments M_b and M_t are obtained by use of moment equilibrium. The cross-section at φ is selected for moment equilibrium. Then the shear force (not shown in the figure) does not enter into the equilibrium equations. One obtains, using x and y as directions for the moment equilibrium:

$$\begin{aligned} &M_b \cos\varphi - M_t \sin\varphi - M + Qy = 0 \\ &M_b \sin\varphi + M_t \cos\varphi + Q(a - x) = 0 \end{aligned} \qquad (19a, b)$$

Solving for M_b and M_t gives:

$$\begin{aligned} M_b &= -Qy\cos\varphi - Q(a - x)\sin\varphi + M\cos\varphi \\ M_t &= +Qy\sin\varphi - Q(a - x)\cos\varphi - M\sin\varphi \end{aligned} \qquad (20a, b)$$

The elastic strain energy stored in the beam (the full, half-elliptical beam) can now be calculated. One obtains:

$$U^{\text{tot}} = \frac{1}{2EI}\int_0^L M_b^2 \mathrm{d}s + \frac{1}{2GK_t}\int_0^L M_t^2 \mathrm{d}s \qquad (21a)$$

where the unknown bending moment M in the beam (at load P) is included in M_b and M_t, see Equation (20a, b). The integration could be performed over the total length L of the beam (the full length of the half-elliptical beam), but due to the symmetry exploited, it suffices to integrate over half the beam length (i.e. the quarter-elliptical beam). This gives half the total energy stored in the beam. The total amount of energy stored in the beam need not be calculated here. Instead, when using the Castigliano theorem, differentiation with respect to half the force, i.e. $Q = P/2$, may be performed to obtain the same result. This will be done here (this is the reason why $Q = P/2$ was introduced). The boundary conditions of the quarter-elliptical beam are such that symmetry with respect to the xz-plane is maintained. Using $U = U^{\text{tot}}/2$ one obtains:

$$U = \frac{1}{2EI}\int_0^{L/2} M_b^2 \,\mathrm{d}s + \frac{1}{2GK_t}\int_0^{L/2} M_t^2 \,\mathrm{d}s \qquad (21b)$$

where $L/2$ is that part of the beam that is situated in the first quadrant of the Oxy coordinate system.

Using the Castigliano theorem, the slope Θ of the beam at the load Q will be determined. One has $\Theta = \partial U/\partial M$. Also, exploit that $\Theta = 0$, which is obtained because of the symmetry, or alternatively, because M is an interior quantity. It gives:

$$\Theta = 0 = \frac{\partial U}{\partial M} = \frac{1}{2EI}\int_0^{L/2} 2M_b \frac{\partial M_b}{\partial M}\mathrm{d}s + \frac{1}{2GK_t}\int_0^{L/2} 2M_t \frac{\partial M_t}{\partial M}\mathrm{d}s \qquad (22)$$

Enter M_b, $\partial M_b/\partial M$, M_t, and $\partial M_t/\partial M$ from Equation (20) into (22). It gives

$$\begin{aligned} 0 = \frac{1}{EI} &\times \int_0^{L/2} \{-Qy\cos\varphi - Q(a - x)\sin\varphi + M\cos\varphi\} \\ &\times \cos\varphi \mathrm{d}s + \frac{1}{GK_t} \\ &\times \int_0^{L/2} \{Qy\sin\varphi - Q(a - x)\cos\varphi \\ &- M\sin\varphi\}(-\sin\varphi)\mathrm{d}s \end{aligned} \qquad (23)$$

Separate the terms in the integral. Collect terms containing Q on the left hand side and terms containing M, on the right hand side. It gives:

$$\frac{Q}{EI}\int_0^{L/2}\{y\cos\varphi+(a-x)\sin\varphi\}\cos\varphi\,\mathrm{d}s$$
$$+\frac{Q}{GK_t}\int_0^{L/2}\{y\sin\varphi-(a-x)\cos\varphi\}\sin\varphi\,\mathrm{d}s$$
$$=\frac{M}{EI}\int_0^{L/2}\cos^2\varphi\,\mathrm{d}s+\frac{M}{GK_t}\int_0^{L/2}\sin^2\varphi\,\mathrm{d}s \tag{24}$$

Now the three special cases $\beta=0$, $\beta=1$ and $\beta\gg 1$ will be investigated. For the full, half-elliptical beam these three cases correspond to, respectively, a double cantilever beam of length a (two cantilever beams in parallel to each other), a half-circular beam with radius $a=b=R$, and a clamped-clamped straight beam of length $2b=2\beta a$.

The case $\beta=0$ gives $\mathrm{d}y/\mathrm{d}x=0$, $\cos\varphi=0$, $\sin\varphi=-1$, $y=0$, $\mathrm{d}y=0$, and $\mathrm{d}s=-\mathrm{d}x$, which, entered into Equation (24), gives $M=0$. This result was expected, because $\beta=0$ gives that the load P is carried by two cantilever beams (of length a) in parallel to each other, and the two beams are loaded in bending only.

The case $\beta=1$ gives that the ellipse becomes a circle. Enter $a=b=R$ for the radius of the circle. Also, one obtains $y=R\sin\varphi$, $x=R\cos\varphi$, and $\mathrm{d}s=R\mathrm{d}\varphi$. The integration will be performed over the variable φ, and the integration limits become 0 and $\pi/2$. Enter this into (24). It gives:

$$\frac{Q}{EI}\int_0^{\pi/2}\{R\sin\varphi\cos\varphi$$
$$+(R-R\cos\varphi)\sin\varphi\}\cos\varphi\,R\,\mathrm{d}\varphi$$
$$+\frac{Q}{GK_t}\int_0^{\pi/2}\{R\sin\varphi\sin\varphi$$
$$-(R-R\cos\varphi)\cos\varphi\}\sin\varphi\,R\,\mathrm{d}\varphi$$
$$=\frac{M}{EI}\int_0^{\pi/2}\cos 2\varphi\,R\,\mathrm{d}\varphi$$
$$+\frac{M}{GK_t}\int_0^{\pi/2}\sin 2\varphi\,R\,\mathrm{d}\varphi \tag{25}$$

After some simplifications one obtains:

$$\frac{QR^2}{EI}\frac{1}{2}+\frac{QR^2}{GK_t}\frac{1}{2}=\frac{MR}{EI}\frac{\pi}{4}+\frac{MR}{GK_t}\frac{\pi}{4} \tag{26}$$

from which the solution $M=2QR/\pi=PR/\pi$ (for $\beta=1$) is obtained. This solution can be found, for example, in the textbook (solutions manual) [3].

The case $\beta\gg 1$ gives that $\cos\varphi=1$ and $\sin\varphi=0$. Also, using $x/a=\xi$, one obtains:

$$\mathrm{d}s=-\mathrm{d}x\sqrt{1+\frac{\beta^2\xi^2}{1-\xi^2}}=-\mathrm{d}x\sqrt{\frac{1-\xi^2+\beta^2\xi^2}{1-\xi^2}} \tag{27a}$$

As $\beta\gg 1$ the third term in the numerator ($\beta^2\xi^2$) is much larger than the two other terms in the numerator. The two terms may then be neglected. It gives:

$$\mathrm{d}s=-\mathrm{d}x\frac{\beta\xi}{\sqrt{1-\xi^2}} \tag{27b}$$

Enter this, together with $y=\beta a\sqrt{1-\xi^2}$, into Equation (24). It gives:

$$\frac{Q}{EI}\int_a^0\{\beta a\sqrt{1-\xi^2}\cdot 1+0\}$$
$$\times\frac{\beta\xi}{\sqrt{1-\xi^2}}(-\mathrm{d}x)+\frac{Q}{GK_t}\int_a^0 0\,\mathrm{d}x$$
$$=\frac{M}{EI}\int_a^0 1\frac{\beta\xi}{\sqrt{1-\xi^2}}(-\mathrm{d}x)+\frac{M}{GK_t}\int_a^0 0\,\mathrm{d}x \tag{28}$$

Simplification and integration gives:

$$Q\beta a\tfrac{1}{2}=M\left[-\sqrt{1-\xi^2}\right]_0^1=M \tag{29}$$

Thus, $M=Q\beta a/2=Pb/4$ when $\beta\gg 1$.

This result can be verified by studying a cantilever beam of length βa loaded with a force Q and a moment M at the free end. The moment should be such that the slope at the loaded end of the cantilever beam is zero. For the cantilever beam one obtains the slope Θ [1]:

$$\Theta=\frac{Q(\beta a)^2}{2EI}-\frac{M(\beta a)}{EI} \tag{30}$$

Enter $\Theta=0$, and one obtains $M=Q\beta a/2$ in agreement with Equation (29).

For any other value of β it is suitable to solve Equation (24) numerically. As in the statically determinate case studied above, $\cos\varphi$, $\sin\varphi$, y, $\mathrm{d}y$, and $\mathrm{d}s$, as given in Equations (5) to (8), are entered into Equation (28). The integration over $\mathrm{d}s$ then is replaced by integration over $\mathrm{d}x$. Then, remove a^2 and a, respectively, from the integrals in Equation (24) and name the integrals I_3, I_4, I_5, and I_6. These integrals are functions of β only. One obtains:

$$\frac{Qa^2}{EI}I_3(\beta)+\frac{Qa^2}{GK_t}I_4(\beta)=\frac{Ma}{EI}I_5(\beta)+\frac{Ma}{GK_t}I_6(\beta) \tag{31}$$

where Equation (24) together with (5) to (8) give the integrals I_3 to I_6.

Circular cross-section

If the beam cross-section is circular, the relationship $GK_t = EI/1.3$ holds ($\nu = 0.3$) and Equation (31) gives:

$$M = Qa\frac{I_3(\beta) + 1.3I_4(\beta)}{I_5(\beta) + 1.3I_6(\beta)} = f(\beta)Qa \qquad (32)$$

The integrals I_3 to I_6 have been evaluated numerically using the MATLAB package. In Fig. 6 the moment M is given as function of the parameter β. Curve (1) gives M/Qa and Curve (2) gives $M/Q\beta a$. It is seen that the three special cases $\beta = 0$, $\beta = 1$ and $\beta \gg 1$ are obtained (asymptotically in the case $\beta \gg 1$). One finds that $\beta = 0$ gives $M = 0$, $\beta = 1$ gives $M = 2Qa/\pi$, and $\beta \gg 1$ gives $M = Qa/2$ (where $Q = P/2$), as it should.

Now, when the moment M is known, the deflection δ of the beam end can be determined. The Castigliano theorem gives:

$$\delta = \frac{\partial U}{\partial Q} = \frac{1}{2EI}\int_0^{L/2} 2M_b\frac{\partial M_b}{\partial Q}\mathrm{d}s + \frac{1}{2GK_t}\int_0^{L/2} 2M_t\frac{\partial M_t}{\partial Q}\mathrm{d}s \qquad (33)$$

Enter M_b, $\partial M_b/\partial Q$, M_t, and $\partial M_t/\partial Q$ from Equation (20) into (33). Further, enter $\cos\varphi$, $\sin\varphi$, y, $\mathrm{d}y$, and $\mathrm{d}s$ expressed in the variable x, and use the dimensionless variable $\xi = x/a$. It gives:

$$\delta = \frac{Qa^3}{EI}I_7(\beta) + \frac{Qa^3}{GK_t}I_8(\beta) \qquad (34a)$$

where:

$$I_7(\beta) = \int_0^1 \left\{-\frac{y}{a}\cos\varphi - \left(1 - \frac{x}{a}\right)\sin\varphi + f(\beta)\cos\varphi\right\} \times \left\{-\frac{y}{a}\cos\varphi - \left(1 - \frac{x}{a}\right)\sin\varphi\right\} \times \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}\left(\frac{x}{a}\right) \qquad (34b)$$

and

$$I_8(\beta) = \int_0^1 \left\{\frac{y}{a}\sin\varphi - \left(1 - \frac{x}{a}\right)\cos\varphi - f(\beta)\sin\varphi\right\} \times \left\{\frac{y}{a}\sin\varphi - \left(1 - \frac{x}{a}\right)\cos\varphi\right\} \times \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}\left(\frac{x}{a}\right) \qquad (34c)$$

Numerical calculation of the integrals I_7 and I_8 gives the deflection δ according to Figs 7 and 8. In Fig. 7 the deflection has been normalized with respect to Qa^3/EI. It can be seen that the special cases $\beta = 0$ and $\beta = 1$ are regained. For $\beta = 0$ (the case of a straight cantilever beam of length a) one obtains $\delta = Qa^3/3EI - Ma^2/2EI = Qa^3/3EI$ (because $M = 0$ when $\beta = 0$). When $\beta = 1$ it was found above, Equation (3d), that $\delta = 0.2582PR^3/EI = 0.5164Qa^3/EI$.

In Fig. 8 the deflection δ has been normalized with respect to $Q(\beta a)^3/EI$. It can be seen that the special case $\beta \gg 1$ is regained. When β is very large the (half) beam behaves like a straight beam clamped at one end and having sliding boundary condition at the other (the sliding boundary

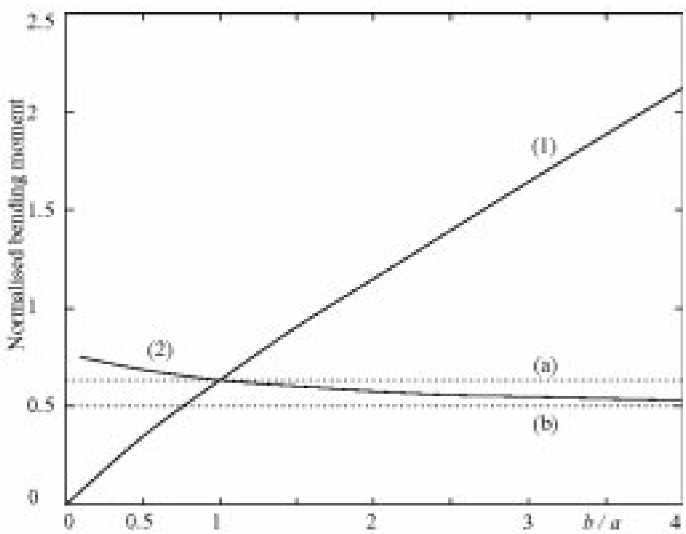


Fig. 6. Bending moment M at symmetry axis (the x-axis, see Fig. 5). Curve (1): Normalised moment M/Qa as function of the parameter $\beta = b/a$, and Curve (2): normalised moment $M/Q\beta a$. Reference line (a) at $2/\pi$ (cf. Equation (26)) and line (b) at 1/2 (cf. Equation (29)).

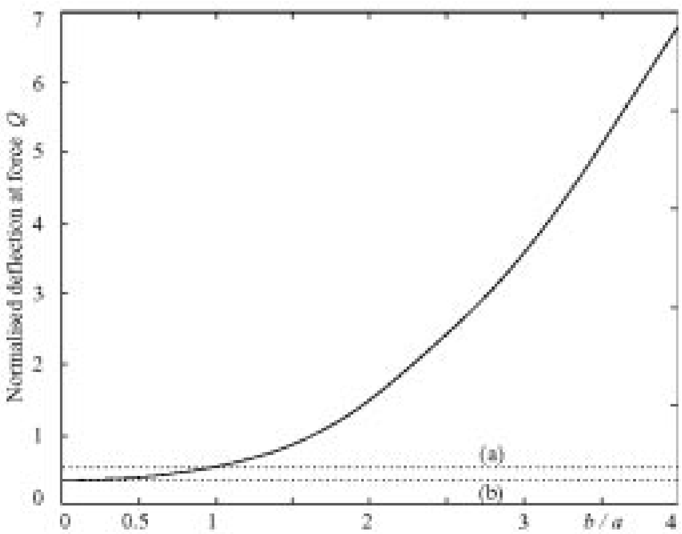


Fig. 7. Deflection $\delta/(Qa^3/EI)$ as function of ratio $\beta = b/a$. Reference line (a) at 0.5164 (from Equation (3d) with $P = 2Q$) and line (b) at 1/3 (from Equation (1)).

condition induces the moment M). A cantilever beam loaded with Q and M gives in this case $\delta = Q(\beta a)^3/3EI - M(\beta a)^2/2EI$ which, using $M = Q(\beta a)/2$, gives $\delta = Q(\beta a)^3/12EI$, as obtained in Fig. 8. Finally, entering $Q = P/2$ one obtains $\delta = P(\beta a)^3/24EI$ as given in expression (3b).

CONCLUSIONS

In this paper, it has been demonstrated that a well known energy method (the Castigliano theorem) can be used, in combination with a numerical integration algorithm, to calculate deflections of curved beams. The curved beams have forms that cannot be found in the handbook literature. Here elliptically curved beams have been investigated. Both statically determinate and statically indeterminate beams have been considered. The loading of the beams is such that the beams are subjected to both bending and torsion at the same time.

If the problems discussed here should be solved with the finite element method, a finite element model had to be created for each ratio of the ellipse's half-axes a and $b = \beta a$, where $0 \leq \beta < \infty$. Not many finite element programs deal with curved beams subjected to both bending and torsion.

In the statically determinate case, the curved beam takes the form of a quarter of an ellipse. The beam is clamped at one end and free at the other. The load, a force P, is applied at the free end perpendicularly to the plane of the curved beam. The beam then is subjected to both bending and torsion. The deflection at the free end is calculated. From an educational point of view, this problem is of interest because there are three limiting cases that can be calculated and compared to results given in handbooks and some textbooks. The half-axes of the elliptically curved beam are a and b

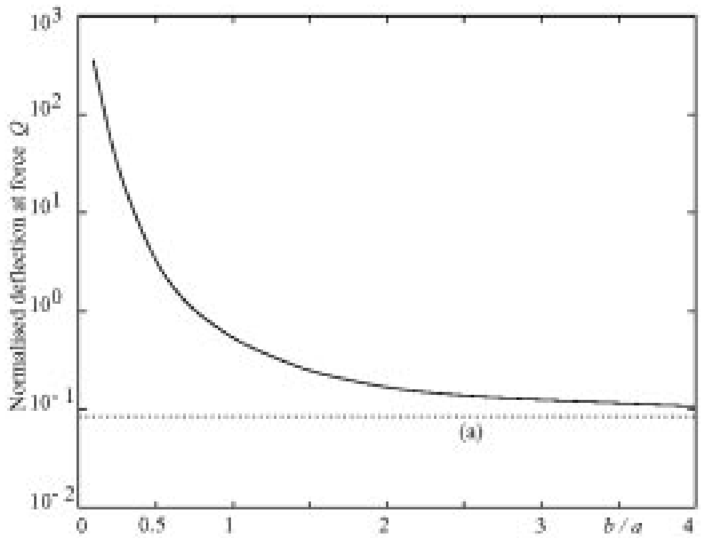


Fig. 8. Deflection $\delta/(Q(\beta a)^3/EI)$ as function of ratio $\beta = b/a$. Reference line (a) at 1/12 (from (3b) with $P = 2Q$).

respectively. In two of the limiting cases, either a or b tends to zero, and the curved beam tends to a straight beam of length b and a, respectively. In the third limiting case, $a=b=R$, the cantilever beam is curved to the form of a quarter of a circle, and this case can be found in some handbooks and a few textbooks.

In the statically indeterminate case, the curved beam takes the form of half an ellipse. The beam is clamped at the two ends and the load, a force P, is applied at the middle of the beam perpendicularly to the plane of the curved beam. This case, thus, is symmetric with respect to the xz-plane, see Fig. 5. Also in this case the beam is subjected to both bending and torsion. The deflection at the load is calculated. Again, from an educational point of view this problem is of interest because also here three limiting cases can be found, and at least two of the limiting cases can easily be found in the handbook literature. These two limiting cases are obtained when either a or b tends to zero. When half-axis b tends to zero, see Fig. 5, the half-ellipse takes the form of two parallel straight cantilever beams of length a. Half the load, $P/2$, is then carried by each one of the two cantilever beams, and the deflection can easily be found in any textbook. When half-axis a tends to zero, the curved beam tends to a straight beam of length $2b$. This beam is clamped at both ends and the load is applied in the middle; a case that can also easily be found in any textbook. The third limiting case appears when $a=b=R$; the beam is then curved to the form of half a circle. This case is, however, less frequent in the handbook literature.

The examples discussed in this paper may be suitable exercises for mechanical and civil engineering students. It is demonstrated how problems, not easily found in the handbook literature, may be solved without too much effort using the second Castigliano theorem and an integration algorithm, for example a MATLAB routine.

REFERENCES

1. J. M. Gere and S. P. Timoshenko, *Mechanics of Materials* (4th edn), PWS Publishing Company, Boston, MA (1997), ISBN 0-534-93429-3.
2. W. C. Young and R. G. Budynas, *Roark's Formulas for Stress and Strain* (7th edition), McGraw Hill, New York, Int. edition (2002), ISBN 0-07-121059-8.
3. T. Dahlberg, *Teknisk hållfasthetslära—Lösningar* (3rd edition), (solutions manual, in Swedish), Studentlitteratur, Lund (2001), ISBN 91-44-02057-0.
4. W. A. Nash, *Schaum's Outlines of Theory and Problems of Strength of Materials* (2nd edition), McGraw-Hill, New York (1987), ISBN 0-07-084366-X.
5. T. Dahlberg, Teknisk *Hållfasthetslära* (3rd edition, in Swedish), Studentlitteratur, Lund (2001), ISBN 91-44-01920-3.

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