The Blackbody Fraction, Infinite Series and Spreadsheets*

DUNCAN LAWSON

*School of Mathematical and Information Sciences, Coventry University, Coventry, CV1 5FB, UK.
E-mail: d.lawson@coventry.ac.uk*

This paper shows how Planck's blackbody equation and the resulting formula for the blackbody fraction can be used as a 'real world' example to motivate undergraduate engineers in their study of mathematics. A spreadsheet can be used to give students familiarity with some of the properties of Planck's spectral distribution. The problem of calculating the blackbody fraction can be solved relatively easily using a spreadsheet provided some simple mathematical analysis is carried out first. The mathematical techniques which are required include change of variable, infinite binomial series, integration by parts and sums of geometric progressions.

INTRODUCTION

ALL ENGINEERS should be fluent in a range of mathematical techniques. However, there is often a motivation problem when teaching mathematics to engineering undergraduates as they fail to appreciate the value of the material they are being taught. One way to address this is through the use of 'real world' applications of the mathematics. Planck's equation for the spectral distribution of the emissive power of a blackbody [1] is an excellent example of a piece of 'real world' mathematics that can illustrate the usefulness of many different mathematical techniques [2]. In this paper we will concentrate on two key aspects:

1. How a simple mathematical transformation can reduce the complexity of a problem.
2. The usefulness of infinite series in allowing evaluation of an otherwise intractable integral.

Planck's equation is not enormously complicated, but it is complex enough that students will gain little knowledge of the nature of the function simply by looking at its formula. They need to spend some time gaining familiarity with the nature of the function before proceeding to some analysis. This familiarity could be gained by using a computer algebra system such as Maple or Mathematica but there are considerable advantages in using a spreadsheet to give students this basic familiarity. In particular, the overwhelming majority of engineering undergraduates are at ease with using a spreadsheet to perform a variety of calculations in other contexts. There is much less widespread fluency with specific mathematical software. Furthermore students are more accepting of results they gain from spreadsheets as they usually have a reasonable grasp of the processes the spreadsheet is implementing. When using a

computer algebra package they often feel unsure about the routines the software has used to achieve the results it presents and therefore feel more insecure about the status of these results.

In what follows we will discuss the basic theory of Planck's equation, the tasks students can undertake to gain familiarity with it and 'discover' for themselves some of its key properties. Then we will progress to a more detailed analysis of the blackbody fraction function and demonstrate how a change of variable (suggested by findings from the earlier stages of investigation) can transform the apparently intractable integral into one that can be evaluated by using infinite series. At this point it is sensible to use a spreadsheet to evaluate truncated versions of this infinite series. Further mathematics related to the sums of infinite series can also be used to determine error bounds for truncating the series after a finite number of terms.

BASIC FAMILIARITY WITH PLANCK'S EQUATION

Planck's equation for the spectral distribution of the emissive power of a blackbody is given by:

$$e_{b\lambda}(\lambda, T) = \frac{2\pi C_1}{\lambda^5(\exp(C_2/\lambda T) - 1)} \qquad (1)$$

In this equation λ is the wavelength, the constants C_1 and C_2 are given by:

$$C_1 = hc^2 = 0.59552197 \times 10^8 \text{ W } \mu\text{m}^4 \text{ m}^{-2} \text{ sr}^{-1}$$

and

$$C_2 = hc/k = 14387.69 \, \mu\text{m K}$$

where h is Planck's constant, $6.6260755 \times 10^{-34}$ J s; k is Boltzmann's constant, 1.380658×10^{-23} J K^{-1} and c is the speed of light in a vacuum 2.97792458×10^8 m s^{-1}.

The Blackbody Function				
C1	59552197	C2	14387.69	
	T1			
	4000			
λ	ebλ(T1)			
0.05	6.85103E-17			
0.1	0.008950385			
0.15	189.8733753			
0.2	18084.63371			
0.25	216213.8369			
0.3	955862.5212			
0.35	2452122.234			
0.4	4544883.228			
0.45	6851348.174			
0.5	9001332.619			
0.55	10756410.52			
0.6	12018920.88			
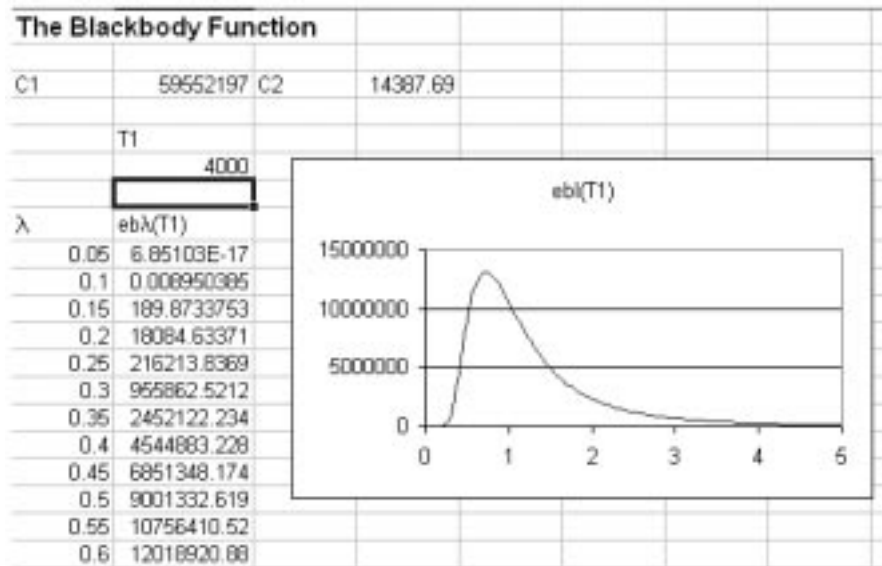
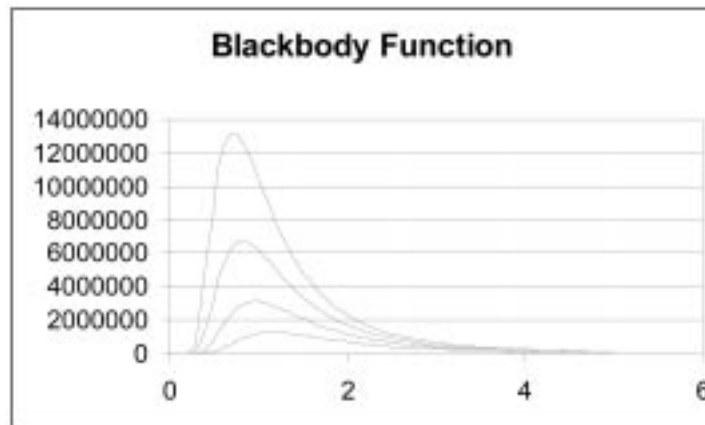Fig. 1. Spreadsheet to plot Planck's blackbody function for a selected temperature.

Fig. 2. The blackbody function for 4000K (highest curve), 3500K, 3000K and 2500K (lowest curve).

A first step towards gaining familiarity with Planck's function is to plot its graph for a number of different temperatures, for example 1000K, 2000K and 4000K. This is easily done in a spreadsheet as shown in Fig. 1.

It is a simple matter to produce a figure showing the graph of the blackbody function for a number of temperatures. Figure 2 shows such a figure for the temperatures 4000K (the curve with the highest peak), 3500K, 3000K and 2500K (the curve with the lowest peak).

The curves drawn all have the same basic shape: they start from zero, rise to a peak and then decay back to zero as the wavelength increases. As the temperature increases the position of the peak moves to the left. It is interesting to note the location of the peak. By looking at the data points used to draw the graphs, a good estimate of the peak wavelength can be obtained. The findings from a selection of temperatures are given in Table 1.

The data points used to plot the graphs are spaced at intervals of $0.05\,\mu$m. For the cases of 1000K and 2000K the points indicated in Table 1 are clearly higher than their neighbours on either side whereas at 4000K the values at 0.7 and $0.75\,\mu$m are almost identical.

The data in Table 1 indicates that as the temperature is doubled so the peak wavelength is halved. In other words $\lambda_p T$ is a constant (where λ_p is the wavelength at which the peak on the $e_{b\lambda}$ curve occurs). This is known as Wein's law [3]. The fact that the location of the peak on the $e_{b\lambda}$ curve does not depend on λ and T independently but only the combination λT suggests that it might be interesting to plot these curves against λT rather than against λ. A spreadsheet which allows two of

Table 1. Wavelength at which the peak value of $e_{b\lambda}$ occurs

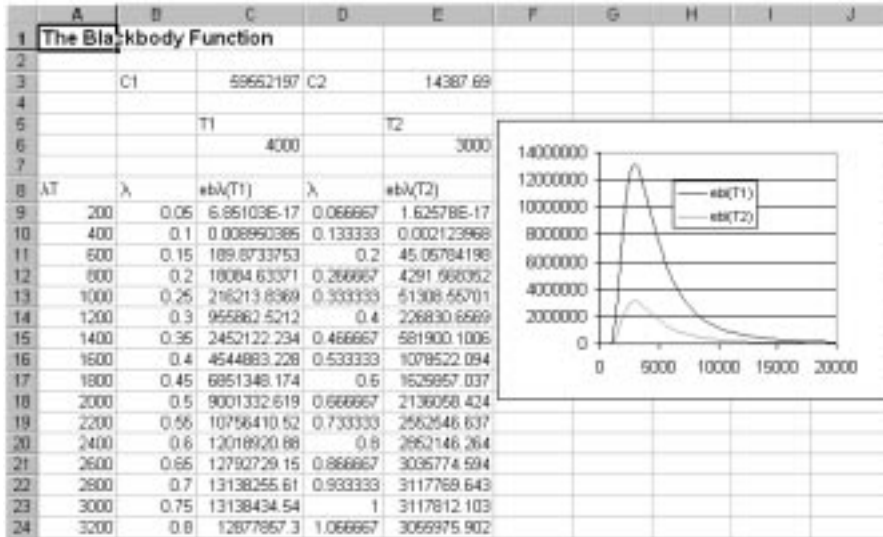| Temperature (K)	Peak wavelength (μm)
1000	2.9
2000	1.45
4000	0.7–0.75

Fig. 3. Spreadsheet to plot $e_{b\lambda}$ curves against λT.

these curves (for different values of T) to be plotted in shown in Fig. 3.

In the spreadsheet shown in Fig. 3, values for the two temperatures are entered in cells C6 and E6. The values in column A are fixed values of λT. The values of λ in column B are calculated from column A by dividing by the value of T_1. Then the values of $e_{b\lambda}$ in column C are calculated from Planck's Equation (1) using the values of λ in column B and the value of T_1. Columns D and E are similar except they use T_2. The graphs are plotted by selecting the data in columns A, C and E and then producing an XY scatter.

The curves confirm what has already been observed, namely that the peak occurs at the same value of λT. Looking at the data used to plot these graphs it is clear that the peak occurs somewhere between λT values of 2800 μm K and 3000 μm K.

The constant location of the peak and the fact that the curves have the same shape suggest that it might be possible to find a scaling which will collapse all the curves onto each other. This is in fact what we observe if we plot $e_{b\lambda}/T^5$ against λT. A spreadsheet to do this is shown in Fig. 4.

This spreadsheet is the same as the one shown in Fig. 3 except that the values in columns C and E have been divided by T_1^5 and T_2^5 respectively. The graph appears to show only one curve, but that is because the values in columns C and E are identical and so the second curve exactly overlays the first.

The above shows that the function $e_{b\lambda}(\lambda,T)/T^5$ is not a function of λ and T independently but a
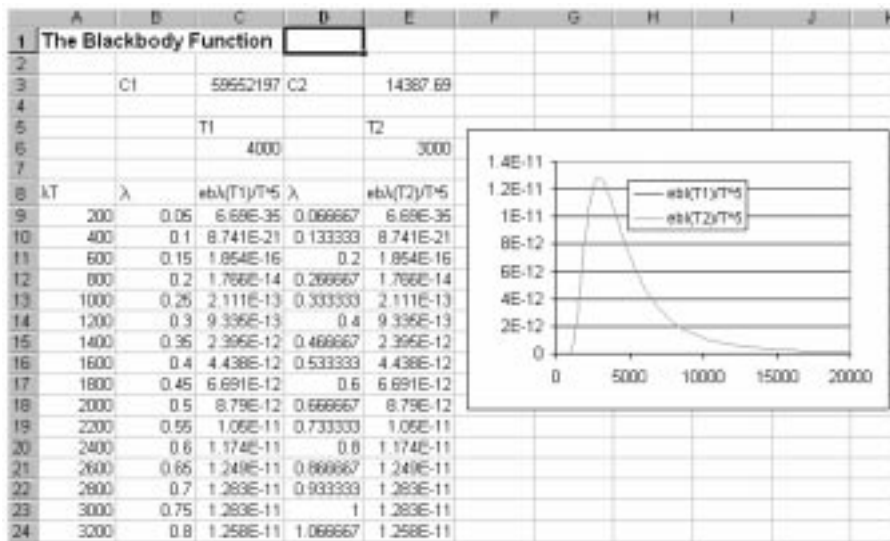

Fig. 4. Spreadsheet to plot $e_{b\lambda}/T^5$ curves against λT.

function of the product λT. This is seen quite easily mathematically by working from Planck's equation (1):

$$\frac{e_{b\lambda}(\lambda, T)}{T^5} = \frac{1}{T^5} \times \frac{2\pi C_1}{\lambda^5(\exp(C_2/\lambda T) - 1)}$$

$$= \frac{2\pi C_1}{(\lambda T)^5(\exp(C_2/\lambda T) - 1)} \quad (2)$$

which shows that we have a function of λT.

THE BLACKBODY FRACTION

The total emissive power is found by integrating the spectral emissive power over the entire spectrum:

$$e_b(T) = \int_0^\infty e_{b\lambda}(\lambda, T)d\lambda = \sigma T^4 \quad (3)$$

where $\sigma = 2\pi^5 k^4/15c^2h^3 = 5.67051 \times 10^{-8}\,\mathrm{W\,m^{-2}\,K^{-4}}$ is the Stefan-Boltzmann constant.

The emissive power in a band of the spectrum, say from λ_1 to λ_2, is found by integrating Equation (1) from $\lambda = \lambda_1$ to $\lambda = \lambda_2$:

$$e_{b,\mathrm{band}}(\lambda_1, \lambda_2, T) = \int_{\lambda_1}^{\lambda_2} e_{b\lambda}(\lambda, T)d\lambda \quad (4)$$

It is clear from the above definition that the emissive power of the band from 0 to λ_2 is the sum of the emissive powers of the bands from 0 to λ_1 and from λ_1 to λ_2. Therefore we have:

$$e_{b,\mathrm{band}}(\lambda_1, \lambda_2, T) = e_{b,\mathrm{band}}(0, \lambda_2, T)$$
$$- e_{b,\mathrm{band}}(0, \lambda_1, T) \quad (5)$$

From Equation (5) we see that the emissive power of a band anywhere in the spectrum can be calculated if we know the emissive power of each band starting from zero wavelength.

We define the blackbody fraction, $F(0, \lambda, T)$, of the wavelength band from 0 to λ, to be the fraction of blackbody power in the band. From Equations (1), (3) and (4) this gives:

$$F(0, \lambda, T) = \frac{e_{b,\mathrm{band}}(0, \lambda, T)}{e_b(T)}$$

$$= \frac{1}{\sigma T^4} \int_0^\lambda \frac{2\pi C_1}{\lambda^5(\exp(C_2/\lambda T) - 1)} d\lambda \quad (6)$$

There is no obvious closed form for the integral on the right-hand side of Equation (6) and so some computational method is needed to evaluate it. Jain [4] used the numerical integration routines within Mathematica to calculate some blackbody fractions, these are shown in Table 2.

Although the use of a sophisticated piece of mathematical software is appealing it has some disadvantages. Primarily the computed answers are not particularly accurate (as will be shown later). No doubt with suitable choice of options

Table 2. Blackbody fractions in selected bands from Jain [4]

Temperature (K)	Band 0–0.38 μm	Band 0–0.76 μm
2500	0.00017	0.05209
5000	0.05129	0.44295
10 000	0.44342	0.83909

within the package more accurate values could be calculated but as there is no indication of the error in these values it would be difficult to be certain when an accurate value had been achieved.

The complexity of the integral can be reduced by making a simple mathematical transformation. This illustrates the value of carrying out some mathematical analysis before rushing into evaluation.

AN ALTERNATIVE APPROACH TO CALCULATING THE BLACKBODY FRACTION

We have already seen that in the theory of spectral radiative emission the combination of the variables λ and T into a single variable λT sometimes has significance. This gives an indication of a way to proceed with the integral in Equation (6). Before we do that we need to restate Equation (6) in a more mathematical precise manner. The way it is given in Equation (6) is how it is presented in many expositions of the theory, however it is mathematically sloppy. The symbol λ is being used in two distinctly different ways. Firstly it specifies one of the arguments of the function F and it is in this capacity that it appears as the upper limit on the integral. However it also appears as the integration variable in the integrand. So, for example, if we wished to calculate the blackbody fraction for the interval from 0 to 0.5 μm ($F(0, 0.5, T)$) we would integrate from 0 to 0.5 but leave all the λ's in the integral unchanged. To avoid this confusing double use of the symbol λ, we will restate Equation (6) using ν as the integration variable:

$$F(0, \lambda, T) = \frac{e_{b,\mathrm{band}}(0, \lambda, T)}{e_b(T)}$$

$$= \frac{1}{\sigma T^4} \int_0^\lambda \frac{2\pi C_1}{\nu^5(\exp(C_2/\nu T) - 1)} d\nu \quad (7)$$

At this point it appears that the blackbody fraction depends on λ (in the upper limit of the integral) and T (in the denominator of the coefficient of the integral and the denominator of the exponent in the integrand) completely independently. However, following the approach of Chang and Rhee [5], we take the T inside the integral sign to obtain:

$$F(0, \lambda, T) = \frac{1}{\sigma} \int_0^\lambda \frac{2\pi C_1}{(\nu T)^5(\exp(C_2/\nu T) - 1)} T d\nu$$

$$(8)$$

Fig. 5. Spreadsheet for calculating the blackbody fraction.

Changing to an integration variable of νT, we can write equation (8) as:

$$F(0, \lambda, T) = \frac{1}{\sigma} \int_0^{\lambda T} \frac{2\pi C_1}{(\nu T)^5 (\exp(C_2/\nu T) - 1)} d(\nu T)$$

$$= \frac{1}{\sigma} \int_0^{\lambda T} \frac{2\pi C_1}{x^5 (\exp(C_2/x) - 1)} dx \qquad (9)$$

Equation (9) shows clearly that $F(0, \lambda, T)$ does not depend on λ and T independently, but only on the product λT (which only appears in the upper limit of the integral). In recognition of this, henceforth we shall refer to $F(0, \lambda, T)$ as $F(\lambda T)$.

Further progress can be made by making a further change to the integration variable, setting $z = C_2/x$, this leads to:

$$F(\lambda T) = \frac{2\pi C_1}{\sigma C_2^5} \int_\infty^{C_2/\lambda T} \frac{z^5}{(\exp(z) - 1)} \left(\frac{-C_2}{z^2}\right) dz$$

$$= \frac{15}{\pi^4} \int_{C_2/\lambda T}^\infty \frac{z^3 e^{-z}}{1 - e^{-z}} dz \qquad (10)$$

where the definitions of C_1, C_2 and σ given earlier have been used to simplify the coefficient of the integral and the numerator and denominator of the quotient in the integrand have been multiplied by e^z to enable expansion of the denominator using the binomial theorem.

The factor $(1 - e^{-z})^{-1}$ in the integrand can be written as an infinite series using the binomial expansion $(1 - x)^{-1} = 1 + x + x^2 + x^3 + \ldots$ which converges for $|x| < 1$. Since all values of z in the range of integration are positive we have $|e^{-z}| < 1$ and so the expansion can be used. This leads to:

$$F(\lambda T) = \frac{15}{\pi^4} \int_{C_2/\lambda T}^\infty z^3 e^{-z} (1 + e^{-z} + e^{-2z} + \ldots) dz$$

$$= \frac{15}{\pi^4} \sum_{n=1}^\infty \int_{C_2/\lambda T}^\infty z^3 e^{-nz} dz \qquad (11)$$

Each of the integrals on the right hand side of equation (11) can be integrated using repeated integration by parts giving:

$$F(\lambda T) = \frac{15}{\pi^4} \sum_{n=1}^\infty \frac{e^{-nz}}{n} \left(z^3 + \frac{3z^2}{n} + \frac{6z}{n^2} + \frac{6}{n^3}\right)$$

$$= \frac{15}{\pi^4} \sum_{n=1}^\infty c_n(z) e^{-nz} \qquad (12)$$

where $z = C_2/\lambda T$.

We can now use a spreadsheet to calculate approximate values for $F(\lambda T)$ by truncating the infinite series in Equation (12) after a finite number of terms. Figure 5 shows a possible spreadsheet implementation of the required calculation.

The values of T and λ are entered in, respectively, cells B3 and D3. The value of z is then calculated in cell D5. An explanation of the other columns is given below:

- the column labelled 'N' gives the term number;
- the column labelled 'Coeff' calculates the values of the coefficients $c_n(z)$ from equation (12);
- the column labelled 'Term' calculates the Nth term in the sum (namely $c_n(z) e^{-nz}$);
- the column labelled 'Sum' calculates the sum of the first N of the terms;
- the column labelled 'F(N)' calculates the approximation to $F(\lambda T)$ obtained by truncating the sum in equation (12) after N terms (i.e. the value of 'Sum' multiplied by $15/\pi^4$).

We can see from Fig. 5 that only three terms are needed to evaluate the blackbody fraction when $T = 5000$K and $\lambda = 0.76 \, \mu$m (i.e. $F(3800)$) correct to 6 decimal places.

From Table 2 we see that Jain [4] calculated this value to be 0.44295. The fact that the sum appears to have converged (as indicated by the unchanging values in the column F(N)) gives us confidence that the value calculated by the spreadsheet is more accurate. A further indication of the inaccuracies

in Jain's values is the fact that the value quoted for the case $T = 10000\,K$ and $\lambda = 0.38\,\mu m$ is 0.44342. Since with this combination of values $\lambda T = 3800\,\mu m\,K$, the value of the blackbody fraction in these two cases must be the same.

Whilst the above may give us confidence that values obtained by truncating the infinite series are more accurate than those obtained by using numerical integration, it would be better to be able to state an upper limit in the error. This can be obtained by using some elementary results about the convergence of geometric series where the common ratio is less than 1. We let $E(N)$ denote the error obtained by truncating the infinite series in Equation (12) after N terms. Then we have:

$$E(N) = \frac{15}{\pi^4} \sum_{n=N+1}^{\infty} c_n(z)e^{-nz} < \frac{15}{\pi^4} c_{N+1}(z) \sum_{n=N+1}^{\infty} e^{-nz}$$

(13)

To obtain this inequality we have observed that, for a given value of z, the sequence of coefficients $c_n(z)$ is strictly monotonic decreasing and so all the coefficients in the exact expression for $E(N)$ (after the first one) are less than $c_{N+1}(z)$. The sum that remains in inequality (13) is simply the sum of a geometric progression with first term $e^{-(N+1)z}$ and common ration e^{-z}. Since this value is less than 1 we can calculate the sum to infinity of this geometric progression to get:

$$E(N) < \frac{15}{\pi^4} c_{N+1}(z) \frac{e^{-(N+1)z}}{1 - e^{-z}}$$

(14)

The spreadsheet for calculating the blackbody fraction is easily extended to include this error bound, as shown in Fig. 6. The formula used to calculate the error bound of truncating after N terms uses the coefficient value from the next row. It can be seen that the values of the coefficients decreases gradually and so the error bound, whilst conservative, is not unduly so.

Table 3. Blackbody fraction values calculated using the truncated infinite series

λT (μm K)	Possible λ and T values	$F(\lambda T)$
950	$\lambda = 0.38\,\mu m$ $T = 2500$ K	0.000174
1900	$\lambda = 0.38\,\mu m$ $T = 5000$ K or $\lambda = 0.76\,\mu m$ $T = 2500$ K	0.052110
3800	$\lambda = 0.38\,\mu m$ $T = 10000$ K or $\lambda = 0.76\,\mu m$ $T = 5000$ K	0.443376
7600	$\lambda = 0.76\,\mu m$ $T = 10000$ K	0.839068

From the values of T and λ used in Table 2 we can see that there should only be four different values of the blackbody fraction (for $\lambda T = 950$, 1900, 3800 and 7600). The values calculated using the spreadsheet implementation of the infinite sum are given in Table 3.

All the values in Table 3 are accurate to 6 decimal places. In order to obtain this accuracy in the last case it was necessary to include 6 terms in the truncated sum. In general, as the value of λT increases so will the number of terms required to obtain a given accuracy (since the value of z is decreased thereby reducing the size of e^{-z} in the error bound).

CONCLUSION

We have shown how Planck's equation for the spectral distribution of the emissive power of a blackbody can be used as a 'real world' application of a wide range of mathematical techniques including change of variable, integration by parts, binomial expansions, sum to infinity of geometric progressions. Furthermore, we have shown that some careful applications of these techniques turns a problem which appears to need sophisticated mathematical software to solve it into one that can easily be solved using a spreadsheet.

	A	B	C	D	E	F
1	**Blackbody Fraction Function**					
2						
3	Temp (K)	5000	λ (μm)	0.76		
4						
5	C2	14387.69	z=C2/(λT)	3.786234		
6						
7	N	Coeff	Term	Sum	F(N)	Error Bound
8	1	126.0019	2.857831	2.857831	0.440077	0.003331744
9	2	41.10526	0.021145	2.878977	0.443333	4.37286E-05
10	3	23.78659	0.000278	2.879254	0.443376	6.93643E-07
11	4	16.63577	4.4E-06	2.879259	0.443376	1.20739E-08
12	5	12.76717	7.66E-08	2.879259	0.443376	2.22015E-10
13	6	10.35074	1.41E-09	2.879259	0.443376	4.23263E-12
14	7	8.700393	2.69E-11	2.879259	0.443376	8.27826E-14
15	8	7.502543	5.25E-13	2.879259	0.443376	1.65018E-15
16	9	6.593893	1.05E-14	2.879259	0.443376	3.3382E-17
17	10	5.881167	2.12E-16	2.879259	0.443376	6.83248E-19

Fig. 6. Spreadsheet extended to include calculation of the error bound.

REFERENCES

1. M. Planck, Distribution of energy in the spectrum, *Ann. Phys.*, **4**, 1901, pp. 553–563.
2. D. A. Lawson, The black body function, *Teaching Mathematics and its Applications*, **17**(1), 1998, pp. 28–32.
3. R. Seigel and J. R. Howell, *Thermal Radiation Heat Transfer*, 3rd ed., Hemisphere, Washington (1992).
4. P. K. Jain, IR, visible and UV components in the spectral distribution of blackbody radiation, *Phys. Educ.*, **31**, 1996, pp. 149–155.
5. S. L. Chang and K. T. Rhee, Blackbody radiation functions, *Int. Commun. Heat Transfer*, **11**, 1984, pp. 451–455.

Duncan Lawson worked for British Gas following completion of his D.Phil., where he worked in mathematical modelling in both safety and utilisation studies. In 1987 he moved to Coventry University where he has gained extensive experience teaching mathematics to undergraduate engineers. He is co-editor of the SEFI Engineering Mathematics Core Curriculum and is currently Secretary of the SEFI Maths Working Group. He has worked as a consultant for British Gas developing software for calculating radiative heat transfer in complex geometries.