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Efficient Moment Method Solution for the Parallel-Plates Transmission Line Revisited*

M. HARIDIM, H. MATZNER and N. THIRER
Dept. of Communication Engineering, Holon Academic Institute of Engineering, Golomb 52, Holon, Israel. E-mail: mharidim@hait.ac.il

The well known parallel-plates transmission line is solved efficiently by the moment method, where the entire domain expansion functions contain the edge behavior of the fields. It is shown that two expansion functions are enough for an excellent convergence of the solution, in agreement with the analytical conformal mapping solution. Our moment method solution is also compared to other moment method solutions.

AUTHOR QUESTIONNAIRE

1. The paper discusses materials/software for a course in Electromagnetic Engineering (EM fields, microwaves, transmission lines)
2. Students of the following departments are taught in this course: EE, Communication Engineering (RF), Physics.
3. Level of the course (year): 3rd and 4th.
4. Mode of presentation: lecture.
5. The material is part of a regular course.
6. Class or hours required to cover the material: 5–6 hours.
7. Student homework or revision hours required for the materials: 15–20 hours.
8. The novel aspects presented in this paper are: Improvement of the accuracy and the rate of convergence for the solution of the parallel plate transmission line, based on the physical understanding of the problem.
9. The standard textbooks recommended in the course, in addition to author‘s notes**:** N. N. Rao, *Elements of Engineering Electromagnetics,* 5th Ed. Prentice-Hall (2000); R. E. Collin, *Foundations for Microwave Engineering,* 2nd Ed., McGraw-Hill (1992).
10. The material is not covered in the textbooks.

INTRODUCTION

THE PARALLEL-PLATES transmission line (TL) is a very popular educational tool in textbooks in the field of electromagnetic engineering [1, 2]. Due to its relatively simple structure, it often serves also as an instructive example for teaching the method of moments (MM) [1] where the MM is introduced by solving the parallel-plates TL using three pulse basis functions. The solution of Ref. [1] is compared with those obtained by the conventional MM and also by an analytical conformal method [2], which can be used as a reference solution for checking the accuracy and rate of convergence of other methods.

In [3], it has been shown that if the expansion functions used in MM are chosen in accordance with the physical behavior of the fields, both the accuracy and the convergence rate of the solution are significantly improved. Accordingly, this kind of MM may be called the efficient moment method. In this method, the solution is based on the very physical behavior of the electric field, i.e. symmetry and edge conditions, which are known prior to the MM solution. This method has been successfully applied for solving several electrostatic and acoustic problems (see, for example, [5, 6]).

The efficient MM differs from the conventional MM in the choice of the basic expansion functions. In this method the basic expansion functions are composed solely of the known functions describing the true singularity behavior of the electric field. In other words, the efficient MM utilizes the knowledge of irregular field solutions obtained elsewhere, e.g. edges at right angle, in order to facilitate the convergence of the MM solution, whilst yielding a more accurate solution. This is in contrast to the conventional MM that uses arbitrary simple expansion functions, and no attempt is made to ‘guess’ the correct basis functions corresponding to the physical behavior of the electric field in the problem at hand. In addition to its higher accuracy and convergence rate, the efficient MM, being based on the physical behavior of the electric field, also provides a better insight to the solution. In this sense, the efficient MM may be considered a semi-numerical method

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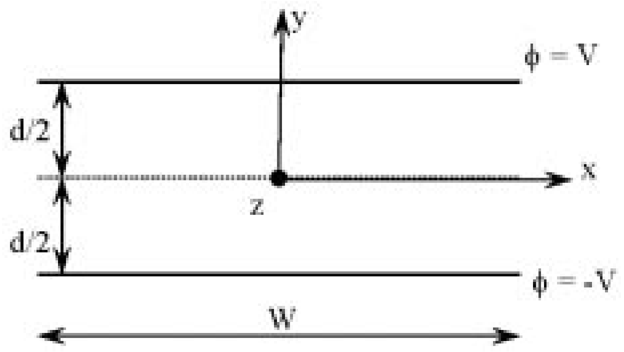
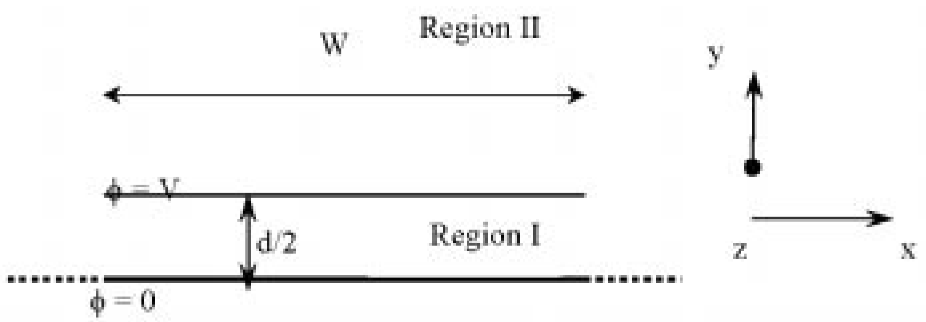


(a)



(b)

Fig. 1. Cross-sectional view of the parallel-plates transmission line. The plates have a width of W and are held at potentials ±V. The distance between the plates is d. (a) the actual line, (b) the equivalent structure.

as compared to the conventional MM which is a pure numerical method. As the efficient MM is based on an understanding of the physical aspects of the problem for selecting the expansion function, it may also be favored for educational purposes.

In this work, we wish to illustrate the solution for the parallel-plates TL using the efficient MM. The solution is based on the Galerkin moment method in which the same set of functions is used as both expansion and test functions. This method provides the students with a valuable insight into physical understanding of the parallel-plates TL.

FORMULATION OF THE MM PROBLEM

The cross-sectional view of the parallel-plate transmission line and the corresponding co-ordinates are shown in Fig. 1a. Figure 1b shows the equivalent structure of the TL used in our analysis. The spacing between the plates is d (y-direction), and their width is W (x-direction). The plates are held at potentials ±V.

The potential function

Following [4], the solution of the Laplace equation:

$$\nabla^2\Phi = 0 \tag{1}$$

for the electric potential in the upper half and above the parallel plates shown in Fig. 1 can be written, respectively, as:

$$\Phi_I(x,y) = \int_0^\infty dk\tilde{V}_I(k)\cos(\mathrm{kx})\sinh(\mathrm{ky}), \mathrm{y} \leq \mathrm{d}/2 \tag{2}$$

$$\Phi_{\mathrm{II}}(\mathrm{x},\mathrm{y}) = \int_0^\infty dk\tilde{V}_{II}(k)\cos(\mathrm{kx})\exp(-\mathrm{ky}), \mathrm{y} \geq \mathrm{d}/2 \tag{3}$$

describing symmetrical functions of x, which obey boundary conditions at $\mathrm{y} \to \infty$ and y = 0.

Requiring the continuity of the potential function at y = $d/2$:

$$\Phi_I(x,d/2) = \Phi_{II}(x,d/2) \tag{4}$$

we have:

$$\tilde{\mathrm{V}}_{\mathrm{II}}(\mathrm{k}) = \frac{1}{2}\tilde{\mathrm{V}}_{\mathrm{I}}(\mathrm{k})[\exp(\mathrm{kd}) - 1] \tag{5}$$

yielding:

$$\Phi(x,y) = \begin{cases} \int_0^\infty dk\tilde{V}_I(k)\cos(\mathrm{kx})\sinh(\mathrm{ky}), & \mathrm{y} \leq \mathrm{d}/2 \\ \int_0^\infty dk\tilde{V}_I(k)\cos(\mathrm{kx})[\exp(\mathrm{kd}) - 1]\exp(-\mathrm{ky}), & \mathrm{y} \geq \mathrm{d}/2. \end{cases} \tag{6}$$

The potential functions in terms of the surface charge density

Applying the boundary conditions for the normal components of the electric fields at the y = d/2 plane

$$\hat{n} \cdot (\vec{D}_{II} - \vec{D}_I) = \rho_s(x) \tag{7}$$

we obtain

$$-\frac{\partial \Phi_{II}(x, d/2)}{\partial y} + \frac{\partial \Phi_I(x, d/2)}{\partial y} = \frac{1}{\varepsilon_0}\rho_s(x) \tag{8}$$

and thus

$$\int_0^\infty dk \tilde{V}_I(k)\cos(kx)[\exp(kd) - 1]\frac{k}{2}\exp\left(-\frac{kd}{2}\right) + \int_0^\infty dk \tilde{V}_I(k)\cos(kx)k\cosh\left(\frac{kd}{2}\right) = \frac{\rho_s(x)}{\varepsilon_0} \tag{9}$$

where $\rho_s(x)$ is the surface charge density on the plate.

Using the Fourier cosine transforms:

$$\tilde{\rho}_s(k) = \int_0^\infty \rho_s(x)\cos(kx)dx \tag{10}$$

$$\rho_s(x) = \frac{2}{\pi}\int_0^\infty \tilde{\rho}_s(k)\cos(kx)dk \tag{11}$$

Equation (9) is rewritten as:

$$\tilde{V}_I(k)\left\{[\exp(kd) - 1]\frac{k}{2}\exp\left(-\frac{kd}{2}\right) + k\cosh\left(\frac{kd}{2}\right)\right\} = \frac{2\tilde{\rho}_s(k)}{\pi\varepsilon_0} \tag{12}$$

and thus:

$$\tilde{V}_I(k) = \frac{2\tilde{\rho}_s(k)}{\pi\varepsilon_0 k}\exp\left(-\frac{kd}{2}\right) \tag{13}$$

The surface charge density

We now apply the boundary condition $\Phi = V$, at the upper plate ($y = d/2$)

$$\int_0^\infty dk\{\tilde{V}_I(k)\cos(kx)\sinh(kd/2) = V, 0 \quad < x < W/2 \tag{14}$$

and substituting $\tilde{V}_I(k)$ from Equation (13), we obtain

$$\int_0^\infty dk \frac{2}{\pi\varepsilon_0 k}\tilde{\rho}_s(k)\exp(-kd/2)\cos(kx)\sinh(kd/2) = V, \quad 0 < x < W/2 \tag{15}$$

yielding:

$$\frac{1}{\pi\varepsilon_0}\int_0^\infty dk \frac{\tilde{\rho}_s(k)}{k}\cos(kx)[1 - \exp(-kd)] = V, \quad 0 < x < W/2 \tag{16}$$

Equation (16) is an integral equation for the unknown function $\tilde{\rho}_s(k)$ the cosine Fourier transform of the charge density on the plates, whose inverse transform yields the charge density function. Since an analytical solution to this integral equation is not known, we apply the MM to solve this equation.

The moment method formulation

The MM solution is based on replacing the unknown function, $\tilde{\rho}_s(k)$, by a linear combination of basis functions:

$$\tilde{\rho}_s(k) = \sum_{j=1}^\infty a_j\rho_j(x) \tag{17}$$

The functions $\{\rho_i(x)\}$ are also called the MM expansion functions, and the unknown coefficients $\{a_j\}$ are called the MM expansion function coefficients.

Choosing the expansion functions

The surface charge density near the edges can be described as:

$$\rho_s(x) \sim \sum_{n=1}^\infty \alpha_n(W/2 - x)^{\beta_n}, \beta_n = n/2 - 1, x \to W/2 \quad \text{and} \quad n = 1, 3, 5, \ldots \tag{18}$$

and

$$\rho_s(x) \sim \sum_{n=1}^\infty \alpha_n(W/2 + x)^{\beta_n}, \beta_n = n/2 - 1, x \to -W/2 \quad \text{and} \quad n = 1, 3, 5, \ldots \tag{19}$$

where α_n are yet arbitrary constants.

As it was mentioned above, the appropriate expansion functions must contain the exact edge behavior of the surface charge density. In addition, the expansion functions should be continuous and of continuous derivatives on the plates. Furthermore, in order to increase the convergence rate of the solution, one should choose expansion functions having an analytical Fourier transform. According to these considerations the following set of expansion functions are chosen [3]

$$\rho_i(x) = \cos^{i-3/2}\left(\frac{\pi x}{W}\right), i = 1, 2, 3, \ldots \tag{20}$$

The MM linear equation system

The last equation is substituted in Equation (16), yielding the MM set of linear equations system:

Table 1. The TL's characteristic impedance [Ù] calculated by various methods, for three values of *d*/*W*.

Method	d/W=10	d/W=1	d/W=0.2
This paper, N=1	442.631896	178.695612	61.922209
This paper, N=2	442.518190	178.061373	58.054936
This paper, N=3	442.518055	178.061255	58.043023
This paper, N=4	442.518055	178.061255	58.043019
Rao [1], N=1	456.103875	188.369126	60.416496
Rao [1], N=10	443.888991	179.203402	58.538541
Rao [1], N=100	442.655629	178.176951	58.096696
Rao [1], N=500	442.545574	178.084417	58.053809
Collin[2]	442.518051	178.056792	58.042994

$$\sum_{j=1}^{N} \mathrm{A_{ij}}\alpha_j = \mathrm{B_i} \tag{21}$$

where N is the number of expansion functions taken for the solution, and

$$A_{ij} = \int_0^{W/2} \left\{ \frac{1}{\pi\varepsilon_0} \int_0^{\infty} \mathrm{dk}\, \frac{\tilde{\rho}_j(\mathrm{k})}{\mathrm{k}} \cos(\mathrm{kx})[1-\exp(-\mathrm{kd})] \right\} \rho_i(x)dx$$
$$= \frac{1}{\pi\varepsilon_0} \int_0^{\infty} \frac{\tilde{\rho}_j(\mathrm{k})}{\mathrm{k}} \tilde{\rho}_i(\mathrm{k})[1-\exp(-\mathrm{kd})]\mathrm{dk} \tag{22}$$

and

$$\mathrm{B_i} = \int_0^{\mathrm{W}/2} \mathrm{V}\rho_i(\mathrm{x})\mathrm{dx} \tag{23}$$

The capacitance per unit length of the transmission line

Having found the MM coefficients, α_j, the charge per unit length on each plate can be calculated by:

$$\mathrm{Q} = \int_{-\mathrm{W}/2}^{\mathrm{W}/2} \left[\sum_{\mathrm{i}=1}^{\mathrm{N}} \alpha_i \rho_i(x) \right] \mathrm{dx} = 2 \sum_{\mathrm{i}=1}^{\mathrm{N}} \alpha_i \tilde{\rho}_i(0) \tag{24}$$

and hence the capacitance per unit length of the transmission line and its characteristic impedance are given by:

$$C = \frac{Q}{V} \tag{25}$$

and

$$\mathrm{Z}_0 = \frac{\sqrt{\mu\varepsilon}}{C} \tag{26}$$

respectively.

RESULTS

The calculations are carried out for a parallel-plate line of $d = W$, as in the case of Ref. [1] so the results can be compared to those obtained by the conventional MM. The results are shown in Table 1 and compared with those obtained by other methods. Since the result of [2] are obtained by the analytical method of conformal transformation it is referred to as an accurate result for the purpose of comparison. Note that the results correspond to an air filled line. For dielectric filled lines, the characteristic impedance must be divided by $\sqrt{\varepsilon_r}$.

As can be seen from Table 1, in the case of the efficient MM only two expansion functions, based on physical behavior of the fields, are sufficient to achieve a highly accurate result of $Z_0 = 178.07\Omega$ as compared to a value of 178.04Ω in Ref. [2]. The small discrepancy between these values can be attributed to the fact that the latter was actually calculated from a less precise result (4 digits, only) for the TL capacitance per unit length. In the case of the conventional MM in which only the symmetrical considerations are used, 10 basic functions are required to achieve a less accurate value of Z_0=179.94Ω.

CONCLUSIONS

In conclusion, it is shown that the efficient MM can be applied to efficiently solve the parallel plates line. Using only two entire-domain edge-type expansion functions, an accurate and fast converging solution has been obtained.

As the method and solution procedures presented in this paper are based on physical interpretation of the problem it provides an excellent educational means for teaching students both the subjects of parallel plates and MM emphasizing the very physical aspects behind the field solution. This method can be used for teaching other electromagnetic problems in a similar fashion.

It should be noted that taking advantage of the known physical behavior of the field is indeed a common practice also in other method. The solution of [1], for instance, is, to a certain extent, simplified by noting the symmetrical geometry of the problem, i.e. the number of expansion functions required to achieve a solution having a given level of accuracy is halved. In our solution, we extended the physical analysis of the problem beyond the symmetry considerations realizing that the exact field solution is closely related to its known behavior at wedges, and hence we have not only reduced the solution complexity (two basic functions instead of 10 in [1]) but also increased its accuracy. The interaction between physical analysis of the problem and the numerical method is a useful educational tool.

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APPENDIX

Fourier transform of the expansion functions

The Fourier transform of the expansion functions is given by:

$$\rho_i(x) = \int_0^{W/2} \cos^{i-3/2}\left(\frac{\pi x}{W}\right)\cos(kx)dx = \frac{W}{\pi} I_c\left(i-3/2, \frac{kW}{\pi}\right)$$

where [9]:

$$I_{ci}(\beta, k) = \int_0^{\pi/2} \cos^{\beta}(t)\cos(kt)dt = \frac{\pi\Gamma(1+\beta)}{2^{1+\beta}\Gamma(1+\beta/2-k/2)\Gamma(1+\beta/2+k/2)}$$

Similarly:

$$B_i = \int_0^{W/2} V\rho_i(x)dx = V\frac{W\Gamma(i/4)}{(i/2-1)\sqrt{\pi}\Gamma(i/4-1/2)}$$

Accelerating the convergence of A_{11}

The integrands of A_{11} behave as k^{-I-j}. In order to accelerate the convergence of A_{11} we write:

$$A_{11} = \frac{1}{\pi\varepsilon_0}\int_0^{\pi/2d} \frac{\tilde{\rho}_1(k)}{k}[1-\exp(-kd)]\tilde{\rho}_1(k)dk + \frac{1}{\pi\varepsilon_0}\int_{\pi/2d}^{\infty} \frac{\tilde{\rho}_{1,as}(k)}{k}\tilde{\rho}_{1,as}(k)dk$$
$$+\frac{1}{\pi\varepsilon_0}\int_{\pi/2d}^{\infty}\left\{\frac{\tilde{\rho}_1(k)}{k}[1-\exp(-kd)]\tilde{\rho}_1(k) - \left[\frac{\tilde{\rho}_1(k)}{k}[1-\exp(-kd)]\tilde{\rho}_1(k)\right]_{as}\right\}dk$$

where the second integral is calculated analytically. We have:

$$I_{\mathrm{c,as}}(\beta,\mathrm{k}) \approx \Gamma(1+\beta)\frac{\sin[\pi(k-\beta)/2]}{k^{1+\beta}} \qquad k \to \infty$$

$$\begin{aligned}
I_{\mathrm{as}} &= \frac{1}{\pi\varepsilon_0}\int_{\pi/2d}^{\infty}\frac{\tilde{\rho}_{1,as}(k)}{k}\tilde{\rho}_{1,as}(k)dk \\
&= \frac{1}{\pi\varepsilon_0}\left(\frac{W}{\pi}\right)^2\int_{\pi/2d}^{\infty}\frac{1}{k}\left\{\Gamma(1/2)\frac{\sin[\pi(kW/\pi+1/2)/2]}{(kW/\pi)^{1/2}}\right\}^2 dk \\
&= \frac{1}{\pi\varepsilon_0}W\int_{\pi/2d}^{\infty}\frac{1}{k^2}\sin^2[kW/2+\pi/4)dk \\
&= \frac{W}{2\pi\varepsilon_0}\int_{\pi/2d}^{\infty}\frac{1}{k^2}[1+\sin(kW)]dk \\
&= \frac{W}{2\pi\varepsilon_0}\left[\frac{4d}{\pi} - WC_i(\pi/2)\right]
\end{aligned}$$

where it is assumed $d{=}W$.

Motti Haridim received a B.Sc.E.E. from Technion Israel, M.Sc.E.E. from University of Washington and a Ph.D. degree in E.E. from Technion Israel (1992). In 1994 he has joined Holon Academic Institute of Technology, HAIT. Since 2002 Dr Haridim is the head of the Dept. of Communication Engineering at HAIT. Dr Haridim is active in the fields of numerical Electromagnetics, microwave photonics, abd optical communications**.**

Haim Matzner finished his B.Sc. studies in the faculty of physics at the Technion—Technology Institute of Israel, Haifa, in 1970. He completed his M.Sc thesis in the faculty of physics in Tel-Aviv University in the department of theoretical elementary particle under the supervision of Prof. J. Dothan in 1977. In 1993 he finished his Ph.D. thesis (In Distinction) in the Department of Physics of Complex Systems in the Weizmann Institute of Science, Rehovot, Israel, under the supervision of Prof. S. Shtrikman, and then joined to same department as a post-doctorant fellow and scientist. In 1997 he joined the Department of Communication Engineering at HAIT, Holon Academic Institute of Technology, Holon, Israel. The main interest subjects of Dr Matzner are antennas, microwaves and efficient solutions to electromagnetic problems.

N. Thirer was born in Jassy, Romania. He received the M.Sc Degree in electrical engineering from the Polytechnic Institute of Jassy, and the Ph.D. Degree in electrical engineering from the Technical University of Jassy, Romania. From 1969 to 1982 he was with the Polytechnic Institute of Jassy, as assistant and lecturer at the Computers and Electronics department and form 1989 with the Holon Academic Institute of Technology, as lecturer at the Electric and Electronics department. He worked also in industry for several years as project engineer for computer based systems at Elscint Ltd and at IDF. His research interests include microprocessor and FPGA based systems, data acquisition systems design, electronic measurements, digital implementation of low vision aids methods He has more than 40 publications in journals and conferences in these areas.