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Planar Vector Equations in Engineering*

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A novel method to solve planar vector equations in two unknowns is outlined and developed and its application is illustrated in the context of kinematic analysis. The problem is classified in terms of all four possibilities concerning the combinations of unknown pairs of vector magnitudes and directions, which may arise in the course of formulation. Through extensive use of 1×2 and 2×1 matrix multiplication and the 2×2 planar, proper orthogonal operator, it is believed that this method offers advantages of simplicity and computational efficiency as well as robustness, compared to conventional approaches.

Keywords: Planar mechanism; vector equation.

INTRODUCTION

MANY ENGINEERING problems are formulated with planar vector equations. Traditionally, these are often solved manually, with graphical construction done to scale or else as a guide to tedious calculations. The use of computers led to the development of a systematic computational approach to solutions which offers certain conceptual advantage as well as the obvious improvement in speed and precision. The most familiar methods used to implement computational solutions include:

- analysis
- complex algebra [1]
- Chace's method [2]

Analytical and complex algebra methods require the solution of simultaneous equations. In the case of nonlinear problems, these often lead to algebraic manipulation. Chace [2] took advantage of the conciseness of vector notation to obtain explicit closed-form solutions to planar vector equations. However, his method involves converting two-dimensional equations into three-dimensional equations and solving the latter. Therefore, the method is very time-consuming. More recently, an orthogonal operator for the synthesis and analysis of some planar robotic manipulators in the plane was introduced [3–7].

A novel way to solve such problems, based on the operator introduced by Angeles [3], will be considered herein. In this method, explicit closed-form solutions to all possible planar vector equations are formulated. Moreover, the planar equations are solved in the plane, not in three dimensions. Therefore, any software implementation of this method will be very efficient. The further advantage of this method arises when one deals with multi-body kinematics or dynamics. In these cases the problems are formulated using matrix computations. This method conveniently can be assembled in matrix form.

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CLASSIFICATION OF PLANAR VECTOR EQUATIONS

Any planar vector equation can be solved for two unknowns. Depending on the unknowns, four distinct cases occur. Consider the vector equation:

$$\mathbf{c} = \mathbf{a} + \mathbf{b} \tag{1}$$

in which $\mathbf{c} = c\hat{\mathbf{c}}$, $\mathbf{a} = a\hat{\mathbf{a}}$ and $\mathbf{b} = b\hat{\mathbf{b}}$, where a, b and c are scalars, representing the magnitudes of the vectors, and $\hat{\mathbf{a}}$, $\hat{\mathbf{b}}$ and $\hat{\mathbf{c}}$ are unit vectors, representing their directions. Then the problem can be classified, according to the unknowns, as:

A. Magnitude and direction of the same vector are the unknowns.
B. The magnitudes of two different vectors are the unknowns.
C. The magnitude of one vector and direction of another one are the unknowns.
D. The directions of two different vectors are the unknowns.

The problem of solving Equation (1) for the unknowns at hand, although apparently linear and quite straightforward, is, in fact quadratic, except for cases A and B. Indeed, constraints of the form $\|\hat{\mathbf{x}}\|^2 - 1 = 0$ must be added to the quadratic Cases C and D, where $\mathbf{x}$ is the vector whose direction is unknown.

SOLUTIONS OF PLANAR VECTOR EQUATIONS

In this section we present the solutions to the planar vector equations by using the operator $\mathbf{E}$, a

2×2 orthogonal matrix which rotates vectors in a plane through a counter-clockwise angle of 90°, i.e.,

$$\mathbf{E} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

A. Magnitude and direction of the same vector are the unknowns, i.e., $\mathbf{c} = c\hat{\mathbf{c}}$ of Equation (1) is sought, which is a trivial case.

B. The magnitudes of two different vectors, let us say a and b, are the unknowns. Then, we post-multiply both sides of Equation (1)-transposed by $\mathbf{E}\hat{\mathbf{a}}$, namely:

$$\mathbf{c}^T\mathbf{E}\hat{\mathbf{a}} = \mathbf{a}^T\mathbf{E}\hat{\mathbf{a}} + \mathbf{b}^T\mathbf{E}\hat{\mathbf{a}} \tag{2}$$

where $\mathbf{E}\hat{\mathbf{a}}$ is perpendicular to $\mathbf{a}$. Therefore, the first term of the right hand side of Equation (2) vanishes. Thus, it can be solved for b as follows:

$$b = \frac{\mathbf{c}^T\mathbf{E}\hat{\mathbf{a}}}{\hat{\mathbf{b}}^T\mathbf{E}\hat{\mathbf{a}}} \tag{3}$$

Similarly, one may post-multiply both sides of Equation (1)-transposed by $\mathbf{E}\hat{\mathbf{b}}$ and solve the resulting equation for a, namely:

$$a = \frac{\mathbf{c}^T\mathbf{E}\hat{\mathbf{b}}}{\hat{\mathbf{a}}^T\mathbf{E}\hat{\mathbf{b}}} \tag{4}$$

C. The magnitude of one vector and the direction of another, say a and $\hat{\mathbf{b}}$, are the unknowns. Then, we post-multiply both sides of Equation (1), transposed by $\mathbf{E}\hat{\mathbf{a}}$, namely:

$$\mathbf{c}^T\mathbf{E}\hat{\mathbf{a}} = a\hat{\mathbf{a}}^T\mathbf{E}\hat{\mathbf{a}} + b\hat{\mathbf{b}}^T\mathbf{E}\hat{\mathbf{a}} \tag{5}$$

As explained earlier, $\mathbf{a}^T\mathbf{E}\hat{\mathbf{a}} = 0$. Then, solving Equation (5) for $\hat{\mathbf{b}}^T\mathbf{E}\hat{\mathbf{a}}$ leads to:

$$\hat{\mathbf{b}}^T\mathbf{E}\hat{\mathbf{a}} = \frac{\mathbf{c}^T\mathbf{E}\hat{\mathbf{a}}}{b} \tag{6}$$

Let us consider $\hat{\mathbf{a}}$, $\mathbf{E}\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ as shown in Fig. 1.

Therefore, Equation (6) leads to:

$$\cos\varphi = \frac{\mathbf{c}^T\mathbf{E}\hat{\mathbf{a}}}{b} \tag{7}$$

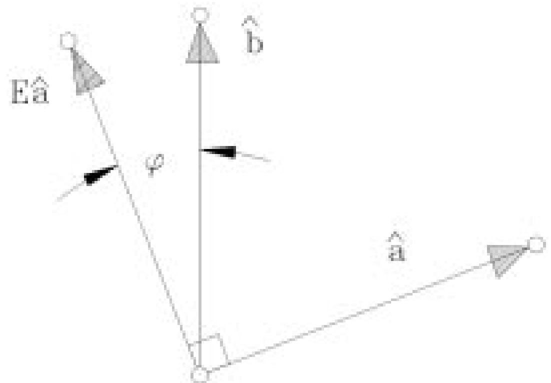

Fig. 1. Unknown direction of **a** and magnitude of **b**.

Thus,

$$\sin\varphi = \pm\sqrt{1 - \cos^2\varphi} \tag{8}$$

Therefore, we have:

$$\hat{\mathbf{b}} = \cos\varphi\mathbf{E}\hat{\mathbf{a}} + \sin\varphi\hat{\mathbf{a}} \tag{9}$$

Finally, one can easily solve Equation (1) for a as:

$$a\hat{\mathbf{a}} = \mathbf{c} - \mathbf{b} \tag{10}$$

D. The direction of two different vectors, let us say $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ are the unknowns. Then, define $\mu = \hat{\mathbf{c}}$ and $\lambda = \mathbf{E}\mu$, as shown in Fig. 2. Therefore, we have:

$$a\hat{\mathbf{a}} = -u\lambda + v\mu \tag{11}$$

$$b\hat{\mathbf{b}} = u\lambda + (c - v)\mu \tag{12}$$

One may compute the magnitudes of vectors **a** and **b** as:

$$(-u)^2 + v^2 = a^2 \tag{13}$$

$$(u)^2 + (c - v)^2 = b^2 \tag{14}$$

Solving Equations (13, 14) for u and v leads to:

$$v = \frac{a^2 - b^2 + c^2}{2c} \tag{15}$$

$$u = \pm\sqrt{a^2 - v^2} \tag{16}$$

Substituting the values of u and v into Equations (13, 14) yields:

$$\hat{\mathbf{a}} = \pm\frac{u}{a}\mathbf{E}\hat{\mathbf{c}} + \frac{v}{a}\hat{\mathbf{c}} \tag{17}$$

$$\hat{\mathbf{b}} = \mp\frac{u}{b}\mathbf{E}\hat{\mathbf{c}} + \frac{c - v}{b}\hat{\mathbf{c}} \tag{18}$$

APPLICATIONS AND EXAMPLES

It is believed that the foregoing methods for solving planar vector equations may be employed to great advantage when applied to the analysis and synthesis of mechanisms and robotic manipulators. In what follows, some examples are presented in support of this contention.

Example 1: Direct kinematics of the slider-crank

The direct kinematics of the slider-crank mechanism of Fig. 3 is the subject of this example. Direct-kinematics problems are those where actuator variables are specified and from these, one seeks to establish the Cartesian co-ordinates of the end-effector (EE). Here, the actuator variable, angle θ_2, is given and the displacement of slider C along the X axis is sought.

The closure vector equation for the displacement is given as:

$$\mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3 = \mathbf{r}_4 \tag{19}$$

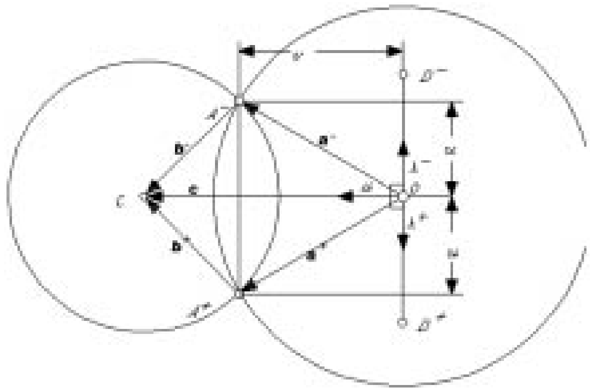

Fig. 2. Two unknown directions.

where $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$ and $\mathbf{r}_4$ are the vectors directed from O to A, A to B, B to C and O to C, respectively.

Equation (19) can be written as:

$$r_4\hat{\mathbf{r}}_4 + (-r_3\hat{\mathbf{r}}_3) = (r_1\hat{\mathbf{r}}_1 + r_2\hat{\mathbf{r}}_2) \tag{20}$$

in which $\{r_i\}_1^4$ and $\{\hat{\mathbf{r}}_i\}_1^4$ are, respectively, the magnitudes and the direction vectors of vectors $\{\mathbf{r}_i\}_1^4$.

In the foregoing equation r_4 and $\hat{\mathbf{r}}_3$ are unknowns. Therefore, this is a class-**C** problem. Thus, the solutions are given by Equations (7–10) through the substitution of the respective given values as follows:

$$\cos\varphi = \frac{(\mathbf{r}_1 + \mathbf{r}_2)^T \mathbf{E}\hat{\mathbf{r}}_4}{-r_3} \tag{21}$$

$$\sin\varphi = \pm\sqrt{1 - \cos^2\varphi} \tag{22}$$

$$\hat{\mathbf{r}}_3 = \cos\varphi\mathbf{E}\hat{\mathbf{r}}_4 + \sin\varphi\hat{\mathbf{r}}_4 \tag{23}$$

$$r_4\hat{\mathbf{r}}_4 = (\mathbf{r}_1 + \mathbf{r}_2) + r_3\hat{\mathbf{r}}_3 \tag{24}$$

Example 2: Slider-crank velocity analysis

In the mechanism of Fig. 3, given the velocity of block C, $\mathbf{v}_C$, find the angular velocities of links AB and BC.

The closure velocity vector equation can be written as:

$$\mathbf{v}_C = \mathbf{v}_B + \mathbf{v}_{C/B} \tag{25}$$

in which $\mathbf{v}_B$ and $\mathbf{v}_{C/B}$ are the velocities of point B and of point C with respect to B, respectively. But we have:

$$\mathbf{v}_B = \omega_2\mathbf{E}\mathbf{r}_2 \tag{26}$$

$$\mathbf{v}_{C/B} = \omega_3\mathbf{E}\mathbf{r}_3 \tag{27}$$

where ω_2 and ω_3 are the angular velocities of links AB and BC, respectively.

Substituting the values of $\mathbf{v}_B$ and $\mathbf{v}_{C/B}$ from Equations (26, 27) into Equation (25) leads to:

$$\mathbf{v}_C = \omega_2\mathbf{E}\mathbf{r}_2 + \omega_3\mathbf{E}\mathbf{r}_3 \tag{28}$$

In Equation (28) the magnitudes of vectors $\omega_2\mathbf{E}\mathbf{r}_2$ and $\omega_3\mathbf{E}\mathbf{r}_3$ are the unknowns. Therefore, this is a class-**B** problem whose solution is outlined above. To find ω_2, we may pre-multiply both sides of Equation (28) by $\mathbf{r}_3^T$, namely:

$$\mathbf{r}_3^T\mathbf{v}_C = \omega_2\mathbf{r}_3^T\mathbf{E}\mathbf{r}_2 + 0 \tag{29}$$

Solving the foregoing equation for ω_2 leads to:

$$\omega_2 = \frac{\mathbf{r}_3^T\mathbf{v}_C}{\mathbf{r}_3^T\mathbf{E}\mathbf{r}_2} \tag{30}$$

Similarly, pre-multiplying both sides of Equation (28) by $\mathbf{r}_2^T$ and solving the resulting equation for ω_3 yields:

$$\omega_3 = \frac{\mathbf{r}_2^T\mathbf{v}_C}{\mathbf{r}_2^T\mathbf{E}\mathbf{r}_3} \tag{31}$$

Example 3: Slider-crank acceleration analysis

In the mechanism of Fig. 3, if the acceleration $\mathbf{a}_C$ of block C is given, find the angular accelerations of links AB and BC.

The closure vector equation for acceleration can be written as:

$$\mathbf{a}_C = \mathbf{a}_B + \mathbf{a}_{C/B} \tag{32}$$

where $\mathbf{a}_B$ and $\mathbf{a}_{C/B}$ are the accelerations of point B

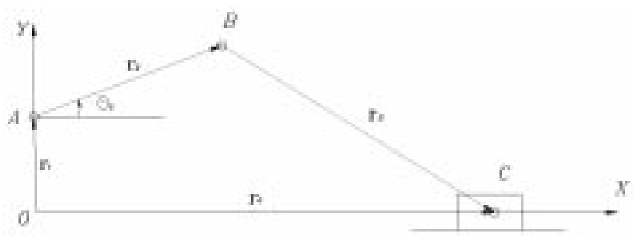

Fig. 3. Offset slider crank mechanism.

and of point C with respect to B, respectively, and can be written as:

$$\mathbf{a}_B = -\omega_2^2\mathbf{r}_2 + \alpha_2\mathbf{E}\mathbf{r}_2 \tag{33}$$

$$\mathbf{a}_{C/B} = -\omega_3^2\mathbf{r}_3 + \alpha_3\mathbf{E}\mathbf{r}_3 \tag{34}$$

in which α_2 and α_3 are the angular accelerations of links AB and BC, respectively.

Substituting the values of $\mathbf{a}_B$ and $\mathbf{a}_{C/B}$ from the foregoing equations into Equation (32), upon simplification, leads to:

$$\alpha_2\mathbf{E}\mathbf{r}_2 + \alpha_3\mathbf{E}\mathbf{r}_3 = \mathbf{a}_C + \omega_2^2\mathbf{r}_2 + \omega_3^2\mathbf{r}_3 \tag{35}$$

In Equation (35) the magnitudes of vectors $\alpha_2\mathbf{E}\mathbf{r}_2$ and $\alpha_3\mathbf{E}\mathbf{r}_3$ are the unknowns. Therefore, this is a class-**B** problem, whose solution is outlined above. To find α_2, we pre-multiply both sides of Equation (35) by $\mathbf{r}_3^T$, namely:

$$\alpha_2\mathbf{r}_3^T\mathbf{E}\mathbf{r}_2 = \mathbf{r}_3^T\mathbf{a}_C + \omega_2^2\mathbf{r}_3^T\mathbf{r}_2 + \omega_3^2\mathbf{r}_3^T\mathbf{r}_3 \tag{36}$$

Solving the foregoing equation for α_2 leads to

$$\alpha_2 = \frac{\mathbf{r}_3^T\mathbf{a}_C + \omega_2^2\mathbf{r}_3^T\mathbf{r}_2 + \omega_3^2\mathbf{r}_3^T\mathbf{r}_3}{\mathbf{r}_3^T\mathbf{E}\mathbf{r}_2} \tag{37}$$

Similarly, pre-multiplying both sides of Equation (35) by $\mathbf{r}_2^T$ and solving the resulting equation for α_3 yields:

$$\alpha_3 = \frac{\mathbf{r}_2^T\mathbf{a}_C + \omega_2^2\mathbf{r}_2^T\mathbf{r}_2 + \omega_3^2\mathbf{r}_2^T\mathbf{r}_3}{\mathbf{r}_2^T\mathbf{E}\mathbf{r}_3} \tag{38}$$

Example 4: Velocity analysis of a planar 3-dof, 3-RRR parallel manipulator

A planar 3-dof, parallel manipulator [8], with only revolute joints, is depicted in Fig. 4. Its motors P_1, P_2 and P_3 are fixed to the base. Finding the manipulator Jacobian matrices is the key issue in isotropic design, singularity analysis and dynamic formulation [3, 5]. Herein, we compute the Jacobian matrices of the manipulator under study.

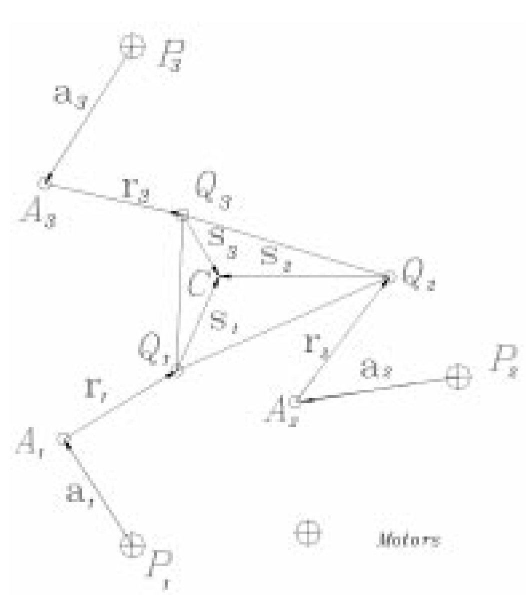

Fig. 4. Planar 3-dof, 3-RRR parallel manipulators.

In planar mechanisms, the relationship between the actuated-joint-velocity vector: $\dot{\theta} = [\dot{\theta}_1, \dot{\theta}_2, \dot{\theta}_3]^T$ and the twist or Cartesian-velocity vector $\mathbf{t} = [\omega \ \mathbf{v}_C]^T$, can be written as:

$$\mathbf{J}\dot{\theta} + \mathbf{K}\mathbf{t} = \mathbf{0} \tag{39}$$

where $\mathbf{J}$ and $\mathbf{K}$ are the two 3×3 Jacobian matrices of the manipulator. Moreover, ω is the angular velocity of the EE.

The velocity $\mathbf{v}_C$ of the EE can be written for the *ith* leg as:

$$\mathbf{v}_C = \mathbf{v}_{A_i} + \mathbf{v}_{Q_i/A_i} + \mathbf{v}_{C/Q_i} \tag{40}$$

where

$$\mathbf{v}_{A_i} = \dot{\theta}_i\mathbf{E}\mathbf{a}_i \tag{41a}$$

$$\mathbf{v}_{Q_i/A_i} = \dot{\gamma}\mathbf{E}\mathbf{r}_i \tag{41b}$$

$$\mathbf{v}_{C/Q_i} = \omega\mathbf{E}\mathbf{s}_i \tag{41c}$$

in which $\dot{\theta}_i$ and $\dot{\gamma}_i$ are the angular velocities of links P_iA_i and A_iQ_i, respectively. Moreover, $\mathbf{a}_i$, $\mathbf{r}_i$ and $\mathbf{s}_i$ are vectors from $\mathbf{P}_i$ to $\mathbf{A}_i$, $\mathbf{A}_i$ and $\mathbf{Q}_i$ to $\mathbf{C}$, respectively.

Substituting the values of $\mathbf{v}_{A_i}$, $\mathbf{v}_{Q_i/A_i}$ and $\mathbf{v}_{C/Q_i}$ from Equations (41a–b) into Equation (40) leads to:

$$\dot{\theta}_i\mathbf{E}\mathbf{a}_i + \dot{\gamma}_i\mathbf{E}\mathbf{r}_i + \omega\mathbf{E}\mathbf{s}_i - \mathbf{v}_C = \mathbf{0} \tag{42}$$

where $\dot{\gamma}_i$, the velocity of an unactuated joint, should be eliminated. This can be done by multiplying Equation (42) by $\mathbf{r}_i^T$, namely:

$$\mathbf{r}_i^T\dot{\theta}_i\mathbf{E}\mathbf{a}_i + \mathbf{r}_i^T\omega\mathbf{E}\mathbf{s}_i - \mathbf{r}_i^T\mathbf{v}_C = 0 \tag{43}$$

Writing the above equation for $i = 1, 2, 3$, we obtain:

$$\mathbf{J}\dot{\theta} + \mathbf{K}\mathbf{t} = \mathbf{0} \tag{44}$$

The 3×3 Jacobian matrices $\mathbf{J}$ and $\mathbf{K}$ are given as:

$$\mathbf{J} \equiv \begin{bmatrix} \mathbf{r}_1^T\mathbf{E}\mathbf{a}_1 & 0 & 0 \\ 0 & \mathbf{r}_2^T\mathbf{E}\mathbf{a}_2 & 0 \\ 0 & 0 & \mathbf{r}_3^T\mathbf{E}\mathbf{a}_3 \end{bmatrix} \tag{45a}$$

$$\mathbf{K} \equiv \begin{bmatrix} \mathbf{r}_1^T\mathbf{E}\mathbf{s}_1 & -\mathbf{r}_1^T \\ \mathbf{r}_2^T\mathbf{E}\mathbf{s}_2 & -\mathbf{r}_2^T \\ \mathbf{r}_3^T\mathbf{E}\mathbf{s}_3 & -\mathbf{r}_3^T \end{bmatrix} \tag{45b}$$

Using the $\mathbf{E}$ operator, the Jacobian matrices are computed based on invariants of the manipulator. This can find vast applications in velocity, dynamic and singularity analyses as well as in the isotropic designs of robotic manipulators.

CONCLUSIONS

A novel method to solve all possible planar vector equations was outlined. It has been proven that the method is applicable to both simple mechanisms and complex robotic manipulators. The simplicity, computational efficiency, and robustness of the method were more obvious when one dealt with robotic manipulators in which the analysis required matrix computations.

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