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# Simplified Formulation of Solution for Beams on Winkler Foundation allowing Discontinuities due to Loads and Constraints*

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*The paper discusses material for a course in Structural Mechanics addressed to second-year Civil Engineering students. The response of beams on a Winkler foundation characterized by discontinuities in both the displacements (deflections and/or slopes) and forces (internal forces and/or loads) is studied. In particular, a simplified formulation for the solution of the discontinuous differential equation governing this problem is given. In some cases, the formulation is able to give the exact solution in a closed form. This is made possible through the use of the generalized functions, such as the well-known Unit Step Function and the Dirac delta function. The cases of discontinuities due to both loads and constraints are treated. The method of presentation of this material is by lecture. The time required to cover the arguments is 2 to 3 hours with 1 to 2 hours of revision. The lecture must be given after the classical beam theory has been covered. The new formulation presented in this paper will demonstrate that some particular aspects of mathematician analysis may be used to advantage to simplify an important problem in structural mechanics.*

**Keywords:** Winkler foundation; generalized functions; discontinuous beam solution.

## NOMENCLATURE

$EI$ beam bending stiffness
$F_i$ concentrated external force
$L[\bullet]$ Laplace transform of the $\bullet$ function
$M_i$ concentrated external moment
$M, V$ bending moment and shear internal forces
$R_n(x - x_0)$ $n$-th order ramp functions.
$R_0(x - x_0)$, $R_1(x - x_0)$, $R_2(x - x_0)$, 0-th order ramp (*unit step*), 1-st order ramp (*unit linear ramp*), and 2-nd order ramp (*parabolic ramp*)
$\bar{R}_0(x, x_i, a_i)$ window function
$u(x)$ beam deflection
$x, s$ axis abscissa and its Laplace transform.
$c_i$ integration constant
$p(x)$ distributed vertical load
$q_i(x)$ distributed vertical load acting on a portion of the beam
$u_0(x)$, $u_p(x)$ homogeneous and particular solution for beam deflection
$\Delta\widehat{\varphi}$ relative slope
$\Delta\widehat{u}$ relative deflection
$\alpha$ relative soil/beam stiffness parameter
$\delta(x - x_0)$ unit impulse function
$\varphi(x)$ beam slope
$\rho_{i,j}, f_{i,k}$ unknown discontinuity in beam differential equation due to along axis beam constraints and concentrated external loads, respectively.

## SIGN CONVENTION

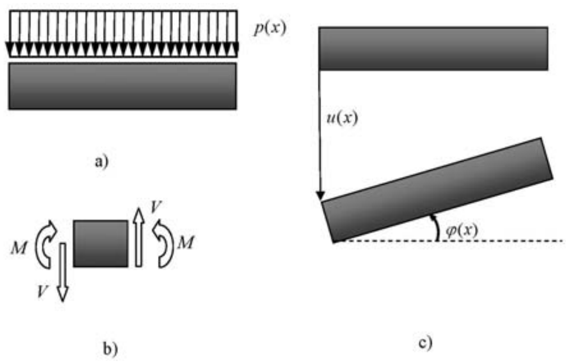


Fig. 1. Sign convention for: (a) applied transverse distributed loads; (b) internal forces; (c) displacements.

## INTRODUCTION

ELASTIC BEAMS under bending actions resting on an elastic foundation and loaded by forces, bending moments and distributed loads are of great importance in applied mechanics mainly because of their practical applications in civil engineering. Many structural elements related to

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soil–structure interaction can be modeled through this scheme, such as railroad tracks, highway pavements, continuously supported pipelines and strip foundations. Various types of foundation models (such as those due to Winkler, Pasternak, Vlasov, Filonenko-Borodich, etc.) have been used in the analysis of structures on elastic foundations [1]. Among these, the Winkler model, in which the terrain is taken into account as a system composed of infinitely close linear springs [2], is the simplest and a frequently adopted one. It assumes that the foundation applies only a reaction normal to the beam axis that is proportional to the beam deflection. Thanks to its simplicity, the Winkler model has been extensively used to solve soil–foundation–structure interaction problems with satisfactory results for many practical problems. Among the many solutions of beams on elastic Winkler type foundation, Hetenyi [3] provided the classical solution of a fourth order governing differential equation for a beam of uniform section. This solution is particularly simple to determine when all the displacements (deflections and slopes) and forces (internal forces and loads) involved in the problem are continuous along the whole beam axis. Otherwise, the only way to apply this method is to divide the beam axis into a number of portions such that in each portion these quantities are continuous. Hence, the evaluation of the exact response can be achieved only by finding the solution of a system of $n$ fourth order differential equations, requiring $4n$ boundary conditions (essential and/or natural).

Alternatively, some approximate methods have been considered for the problem of the beam on elastic foundations, such as: the finite element approach [4], the finite difference method [5], the so-called differential quadrature method, introduced by Chen [6], the approach based on the use of Green's function formulation [7].

These numerical methods can be very effective in the practice. Nevertheless, the presence of an analytical solution of a problem is very important both from a theoretical point of view and from a practical point of view when one wants to verify the accuracy of the results obtained by the numerical approaches.

For this reason, in this paper the problem of a beam on a Winkler-type foundation, in which one or more displacements and/or forces are discontinuous, will be treated by an analytical formulation based on the solution of only one fourth order differential equation, requiring an extremely reduced number of boundary conditions. It will be pursued by extending an approach used by some authors [8–10] for solving the classical beam-bending differential equations, characterized by discontinuities, through the use of the so-called generalized functions [11]. For example, one of the most used generalized functions in any field of sciences is the Dirac delta function [12] and all the other generalized function used in this framework are its derivatives or integrals, to be considered in the generalized sense [11]. In [10] Falsone showed that, when the deflection function of an elastic Bernoulli beam have to be evaluated for any kind of discontinuity, it is always possible to write only one fourth order beam bending differential equation: a considerable advantage from a computational point of view. Here, the same approach will be extended to the governing differential equation of a beam with constant bending stiffness and any kind of discontinuities resting on a Winkler elastic foundation. Finally, it will be shown that, in some simple cases, this approach is able to give the exact closed form of the solution.

## PRELIMINARY CONCEPTS

The differential equation governing the deflection, $u(x)$, of a homogeneous elastic bending beam with constant bending stiffness resting on a Winkler foundation and subjected to a transversal continuous load $p(x)$ can be written as [3]:

$$EIu''''(x) + ku(x) = p(x) \tag{1}$$

where $EI$ is the constant bending stiffness of the beam and $k$ is the elastic foundation modulus. Equation (1) is a continuous differential equation whose general solution $u(x)$ is the sum of the solution $u_0(x)$ of its homogeneous part and of a particular solution $u_p(x)$. The solution $u_0(x)$ has the following form:

$$u_0(x) = \exp(\alpha x)[c_1 \cos(\alpha x) + c_2 \sin(\alpha x)] + \exp(-\alpha x)[c_3 \cos(\alpha x) + c_4 \sin(\alpha x)] \tag{2}$$

where $\alpha = \sqrt[4]{k/(4EI)}$. The expression of the particular solution $u_p(x)$ depends on the load $p(x)$ type. For example, if the load is constant, then $u_p$ is constant too, and given by $u_p = p/k$. The general solution $u(x)$ is completely defined once that the constants $c_i$ are evaluated by imposing the natural and essential boundary conditions. Unfortunately, in many cases, the beams on Winkler foundations are characterized by the presence of discontinuities, due to loads and/or constraints, making the formulation for the solution more complicated. In fact, in these cases, the procedure usually adopted in the literature consists of dividing the beam into $n$ parts, in such a way that in each part of the solution can be considered as continuous. This implies the necessity of writing $n$ solutions as in Equation (2), and hence the necessity of evaluating $n \times 4$ integration constants by imposing $n \times 4$ essential and natural boundary conditions.

The discontinuities can be on the external loads and, as a consequence on the internal forces (bending moment and/or shear), and/or on the displacements (slopes and/or deflections). This last circumstance happens when some internal constraints are along the axis of the beam.

In this paper, it will be shown that the division of the beam into $n$ continuous parts can be avoided by using the so-called generalized functions, already introduced in [10] for solving discontinuous bending beam differential equations. They are reported in [10].

## BEAMS WITH DISCONTINUITIES

The discontinuities in the beam bending differential equations may be due to the external loads when the continuous load is applied only in some portions of the beam and/or when some concentrated forces and/or concentrated moments act on it. Even the presence of along axis constraints (of an essential and/or natural type) determines discontinuities on the governing differential equation and/or in its integrals. In the following, the way in which these discontinuities are taken into account through the use of the generalized functions will be explained.

*Discontinuities due to external loads*

A continuous load $q_i(x)$ acting on a portion of the beam, between the abscissas $x = x_i$ and $x = x_i + a_i$, can be considered as a continuous load along the overall beam axis through the use of the unit step functions, that is:

$$\begin{aligned} p(x) &= q_i(x)[R_0(x - x_i) - R_0(x - x_i - a_i)] \\ &= q_i(x)\bar{R}_0(x, x_i, a_i) \end{aligned} \tag{3}$$

where $\bar{R}_0(x, x_i, a_i)$ is the so-called *window* function, that is the generalized function defined as follows

$$\bar{R}_0(x, x_i, a_i) = R_0(x - x_i) - R_0(x - x_i - a_i) = \begin{cases} 0 & \text{for } x < x_i \\ 1 & \text{for } x_i < x < x_i + a_i \\ 0 & \text{for } x > x_i + a_i \end{cases} \tag{4}$$

Even a concentrated force $F_i$ acting at the abscissa $x = x_i$ can be considered as a load $p(x)$ if the Dirac delta function is introduced, that is:

$$p(x) = F_i\delta(x - x_i) = F_iR_{-1}(x - x_i) \tag{5}$$

At last, a suitable representation of a concentrated moment $M_i$ at $x = x_i$ can be given as:

$$p(x) = M_i\delta'(x - x_i) = M_iR_{-2}(x - x_i) \tag{6}$$

In this way any type of external loads can be represented as a load $p(x)$ acting along the overall beam axis. As consequence, the drawback of considering more partitions of the axis is avoided.

*Discontinuities due to along axis constraints*

As shown in [10], the generalized functions can be usefully considered when any kind of constraint is present at the abscissa $x = x_i$ of the beam axis. For example, if the beam is supported bya roller at $x = x_i$, indicating by $\widehat{F}_i$ the corresponding reaction, then in the beam external loads the following term must be added:

$$\widehat{F}_i\delta(x - x_i) = \widehat{F}_iR_{-1}(x - x_i) \tag{7}$$

The evaluation of the unknown reaction $\widehat{F}_i$ will require the application of the additional essential boundary condition at $x = x_i$, that is $u(x_i) = 0$.

If at $x = x_i$ a constraint on the rotation, that is a double-bearing support, is present then in the beam external loads the following term must be considered:

$$\widehat{M}_i\delta'(x - x_i) = \widehat{M}_iR_{-2}(x - x_i) \tag{8}$$

the moment $\widehat{M}_i$ being the unknown reaction of the support. In this case, the corresponding essential boundary condition is $\varphi(x_i) = -u'(x_i) = 0$.

The presence of hinges or bearing joints along the beam axis implies corresponding discontinuities in the deflections or in the slopes. For example, if a hinge is placed at $x = x_i$, there the slope function must exhibit a discontinuity that is represented by $\Delta\widehat{\varphi}H(x - x_i) = \Delta\widehat{\varphi}R_0(x - x_i)$, where $\Delta\widehat{\varphi}$ is the unknown relative slope. The corresponding boundary condition is essential:

$$M(x_i) = 0 \Rightarrow u''(x_i) = 0 \tag{9}$$

Taking into account that the bending differential equation is of the fourth order in $u(x)$ (Equation (1)), then it must contain the term $-\Delta\widehat{\varphi}\delta''(x - x_i) = -\Delta\widehat{\varphi}R_{-3}(x - x_i)$.

In the case of a bearing join placed at $x = x_i$, the discontinuity is in the deflection function and it can be represented by $\Delta\widehat{u}H(x - x_i) = \Delta\widehat{u}R_0(x - x_i)$, $\Delta\widehat{u}$ being the unknown relative deflection. The corresponding essential boundary condition is related to the shear:

$$V(x_i) = 0 \Rightarrow u'''(x_i) = 0 \tag{10}$$

This implies that the fourth order differential equation must contain the term $\Delta\widehat{u}\delta'''(x - x_i) = \Delta\widehat{u}R_{-4}(x - x_i)$.

*Most general case*

It is important to note that for any type of discontinuity due to along axis constraints, it is always possible to write the governing fourth order differential equation as follows:

$$EIu''''(x) + ku(x) = p(x) + \sum_{i=1}^{N_c}\rho_{i,j}R_j(x - x_i); \quad j = -1, -2, -3, -4 \tag{11}$$

where $N_c$ is the number of along axis beam constraints and $\rho_{i,j}$ are the unknown quantities related to the constraints as follows:

$\rho_{i,-1} \equiv \widehat{F}_i$ roller support
$\rho_{i,-2} \equiv \widehat{M}_i$ double – bearing support
$\rho_{i,-3} \equiv -EI\Delta\widehat{\varphi}_i$ hinge
$\rho_{i,-4} \equiv EI\Delta\widehat{u}_i$ bearing joint.

Finally, in the most general case when any kind of discontinuity is in the beam, the governing differential equation can be written in the following form:

$$
\begin{aligned}
&EIu''''(x) + ku(x) = \sum_{i=1}^{N_q} q_i(x)\bar{R}_0(x, x_i, a_i) \\
&+ \sum_{i=1}^{N_l} f_{i,k} R_k(x - x_i) + \sum_{i=1}^{N_c} \rho_{i,j} R_j(x - x_i); \\
&k = -1, -2; \qquad j = -1, -2, -3, -4
\end{aligned} \tag{12}
$$

where $N_q$ is the number of continuous loads acting on a portion of the beams of length $a_i$, $N_l$ is the number of external concentrated loads acting on the beam and $f_{i,k}$ are the load characteristics defined as follows:

$$
f_{i,-1} \equiv F_i; \qquad f_{i,-2} \equiv M_i \tag{13}
$$

Equation (12) represents the bending differential equation governing the deflection of a beam on Winkler foundation characterized by any kind of external loads and any kind of constraint.

## SIMPLIFIED FORMULATION OF SOLUTION

In this section the approach to carry out a simplified form of the solution of the discontinuous differential equation governing the beams on a Winkler foundation will be shown. In particular, the most general form of this equation, as given in Equation (12), will be taken into account.

The first step consists in rewriting Equation (12) as follows:

$$
\begin{aligned}
&u''''(x) + 4\alpha^4 u(x) = \sum_{i=1}^{N_q} \tilde{q}_i(x)\bar{R}_0(x, x_i, a_i) \\
&+ \sum_{i=1}^{N_l} \tilde{f}_{i,k} R_k(x - x_i) + \sum_{i=1}^{N_c} \tilde{\rho}_{i,j} R_j(x - x_i) \\
&k = -1, -2; j = -1, -2, -3, -4)
\end{aligned} \tag{14}
$$

where $\alpha^4 = k/4EI$, $\tilde{q}_i(x) = q_i(x)/EI$, $\tilde{f}_{i,k} = f_{i,k}/EI$ and $\tilde{\rho}_{i,j} = \rho_{i,j}/EI$. The solution of the last equation can be obtained through the Laplace transform, $L[\bullet]$, that, when applied to both sides of Equation (14), gives:

$$
\begin{aligned}
&\left(s^4 + 4\alpha^4\right) U(s) - \sum_{i=0}^{3} U_i(0) s^{3-i} \\
&= \sum_{i=1}^{N_q} L[\tilde{q}_i(x)\bar{R}_0(x, x_i, a_i)] \\
&+ \sum_{i=1}^{N_l} \tilde{f}_{i,k} s^{-k-1} \exp(-x_i s) \\
&+ \sum_{i=1}^{N_c} \tilde{\rho}_{i,j} s^{-j-1} \exp(-x_i s)
\end{aligned} \tag{15}
$$

where $U_i(0) = \left[\mathrm{d}^i U(s)/\mathrm{d}s^i\right]_{s=0}$ and where it has been considered that the Laplace transform of the generalized function $R_i(x - x_0)$ can be easily obtained as:

$$
\begin{aligned}
L[R_i(x - x_0)] &= \int_0^\infty R_i(x - x_0)\exp(-xs)\mathrm{d}x \\
&= s^{-i-1}\exp(-x_0 s)
\end{aligned} \tag{16}
$$

It is important to note that the Laplace transform $L\left[\tilde{q}_i(x)\bar{R}^{(0)}(x, x_i, a_i)\right]$ into Equation (15) can be easily obtained once that the law of the load $\tilde{q}_i(x)$ has been specified. For example, in the very common case in which it is constant, the following relationship holds:

$$
\begin{aligned}
&L[\tilde{q}_i(x)\bar{R}_0(x, x_i, a_i)] = \tilde{q}_i L[\bar{R}_0(x, x_i, a_i)] \\
&= \tilde{q}_i s^{-1}\exp(-x_i s)[1 - \exp(-a_i s)]
\end{aligned} \tag{17}
$$

The solution of Equation (15) can be written as

$$
\begin{aligned}
U(s) &= \frac{\sum_{i=0}^{3} U_i(0)s^{3-i} + \sum_{i=1}^{N_q} L[\tilde{q}_i(x)\bar{R}_0(x, x_i, a_i)]}{(s^4 + 4\alpha^4)} \\
&+ \frac{\sum_{i=1}^{N_l} \tilde{f}_{i,k} s^{-k-1}\exp(-x_i s) + \sum_{i=1}^{N_c} \tilde{\rho}_{i,j} s^{-j-1}\exp(-x_i s)}{(s^4 + 4\alpha^4)}
\end{aligned} \tag{18}
$$

Then, the inverse Laplace transform is applied, obtaining:

$$
\begin{aligned}
u(x) &= \sum_{i=0}^{3} U_i(0) L^{-1}\left[\frac{s^{3-i}}{s^4 + 4\alpha^4}\right] \\
&+ \sum_{i=1}^{N_q} L^{-1}\left[\frac{L[\tilde{q}_i(x)\bar{R}_0(x, x_i, a_i)]}{s^4 + 4\alpha^4}\right] \\
&+ \sum_{i=1}^{N_l} \tilde{f}_{i,k} L^{-1}\left[\frac{s^{-k-1}\exp(-x_i s)}{s^4 + 4\alpha^4}\right] + \\
&+ \sum_{i=1}^{N_c} \tilde{\rho}_{i,j} L^{-1}\left[\frac{s^{-j-1}\exp(-x_i s)}{s^4 + 4\alpha^4}\right]
\end{aligned} \tag{19}
$$

Now, let us take into account that:

$$
\begin{aligned}
&L^{-1}\left[\frac{s^{-j}}{s^4 + 4\alpha^4}\right] = \frac{\mathrm{d}^{j-1}}{\mathrm{d}x^{j-1}} \\
&\left\{\frac{1}{4\alpha^4}[1 - \cos(\alpha x)\cosh(\alpha x)]\right\} \\
&= \frac{\mathrm{d}^{j-1} g_0(x)}{\mathrm{d}x^{j-1}} = g_{1-j}(x); \\
&j = \cdots - 3, -2, -1, 0, +1, +2 \cdots
\end{aligned} \tag{20}
$$

Hence, we can write, for example:

$$
\begin{aligned}
g_{-4}(x) &= \frac{\mathrm{d}^4 g_0(x)}{\mathrm{d}x^4} = \cos(\alpha x)\cosh(\alpha x) \\
g_{-3}(x) &= \frac{\mathrm{d}^3 g_0(x)}{\mathrm{d}x^3} \\
&= \frac{1}{2\alpha}[\sin(\alpha x)\cosh(\alpha x) + \cos(\alpha x)\sinh(\alpha x)]
\end{aligned}
$$

$$g_{-2}(x) = \frac{d^2 g_0(x)}{dx^2} = \frac{1}{2\alpha^2}[\sin(\alpha x)\sinh(\alpha x)] \tag{21}$$
$$g_{-1}(x) = \frac{dg_0(x)}{dx} = \frac{1}{4\alpha^3}[\sin(\alpha x)\cosh(\alpha x) - \cos(\alpha x)\sinh(\alpha x)]$$
$$g_0(x) = \frac{1}{4\alpha^4}[1 - \cos(\alpha x)\cosh(\alpha x)]$$
$$g_1(x) = \int_0^x g_0(y)\, dy = \frac{1}{4\alpha^4}\left\{x - \frac{1}{2\alpha}[\sin(\alpha x)\cosh(\alpha x) + \cos(\alpha x)\sinh(\alpha x)]\right\}$$

and, due to the translation properties of the Laplace transform, it can be obtained that:

$$L^{-1}\left[\frac{s^{-j}\exp(-sx_i)}{s^4 + 4\alpha^4}\right] = g_{j-1}(x - x_i)R_0(x - x_i) = \bar{g}_{j-1}(x - x_i); \qquad j = \cdots - 3, -2, -1, 0, \cdots \tag{22}$$

The term

$$Q_i(x) = L^{-1}\left[L[\tilde{q}_i(x)\bar{R}_0(x, x_i, a_i)]/(s^4 + 4\alpha^4)\right]$$

can be easily determined when the form of the load $\tilde{q}_i(x)$ is given. For example, if it is constant, then it is not difficult to show that:

$$Q_i(x) = L^{-1}\left[\frac{L[\tilde{q}_i(x)\bar{R}_0(x, x_i, a_i)]}{s^4 + 4\alpha^4}\right] = L^{-1}\left[\tilde{q}_i\frac{s^{-1}}{s^4 + 4\alpha^4}\exp(-x_i s)[1 - \exp(-a_i s)]\right] = \tilde{q}_i[\bar{g}_0(x - x_i) - \bar{g}_0(x - x_i - a_i)] \tag{23}$$

Instead, when the load $\tilde{q}_i(x)$ is linear (with initial value $\tilde{q}_i$ and slope $\Delta\tilde{q}_i$), it can be shown that:

$$Q_i(x) = L^{-1}\left[\tilde{q}_i\frac{s^{-1}\exp(-x_i s)}{s^4 + 4\alpha^4} - (\tilde{q}_i + \Delta\tilde{q}_i)\frac{s^{-1}\exp(-s(x_i + a_i))}{s^4 + 4\alpha^4} + \frac{\Delta\tilde{q}_i}{a_i}\left(\frac{s^{-2}\exp(-x_i s)}{s^4 + 4\alpha^4} - \frac{s^{-2}\exp(-s(x_i + a_i))}{s^4 + 4\alpha^4}\right)\right]$$
$$= \tilde{q}_i\bar{g}_0(x - x_i) - (\tilde{q}_i + \Delta\tilde{q}_i)\bar{g}_0(x - x_i - a_i) + \frac{\Delta\tilde{q}_i}{a_i}(\bar{g}_1(x - x_i) - \bar{g}_1(x - x_i - a_i)) \tag{24}$$

Finally, the general solution of the differential equation is given in the following form:

$$u(x) = \sum_{i=0}^{3} U_i(0)g_{3-i}(x) + \sum_{i=1}^{N_q} Q_i(x) + \sum_{i=1}^{N_l}\tilde{f}_{i,k}\bar{g}_k(x - x_i) + \sum_{i=1}^{N_c}\tilde{\rho}_{i,j}\bar{g}_j(x - x_i) \tag{25}$$

In this equation the four values $U_i(0)$ and the $N_c$ values $\tilde{\rho}_{i,j}$ can be obtained by considering the natural and/or essential conditions at the initial and final abscissas and at the points along the beam axis where constraints act. The number of these unknowns is always equal to $4 + N_c$. On the other hand, if the classical approach is applied, that is partitioning the beam into parts where the solution is continuous, the number of unknowns is equal to $4(N_q + 2 + N_l + 1 + N_c + 1)$. The comparisons between these expressions make evident the advantages of applying the proposed procedure.

Evaluating the solution in terms of slopes, bending moments or internal shear, through the consecutive derivatives of the expression of $u(x)$ given in Equation (25), is trivial.

## THE CLOSED FORM SOLUTION FOR A SIMPLE CASE

The case of a foundation beam of an $n + 1$ bay frame, with distributed vertical loads, concentrated vertical forces and concentrated moments (Fig. 2) can easily be solved in a closed form through the relationships reported in the previous section.

According to Equations (19) and (22) the solution can be written in the following form:

$$u(x) = \sum_{i=0}^{3} U_i(0)g_{3-i}(x) + \sum_{i=1}^{n}\tilde{f}_{i,-1}\bar{g}_{-1}(x - x_i) + \sum_{i=1}^{n}\tilde{f}_{i,-2}\bar{g}_{-2}(x - x_i) + \sum_{j=0}^{n}\tilde{q}_j\left(\bar{g}_0(x - x_j) - \bar{g}_0(x - x_j - a_j)\right) \tag{26}$$

In order to evaluate the integration constants, the four boundary condition at the ends of the beam are imposed, these are the values of the bending moments $M(0) = -EI\,u''(0) = 0$, $M(L) = -EI\,U''(L) = 0$, and of the shear forces $V(0) = -EI\,u'''(0) = 0$, $V(L) = -EI\,u'''(L) = 0$. The conditions at $x = 0$ give two vanishing integration constants $U_2(0) = U_3(0) = 0$. The others two conditions at $x = L$ give two simple linear equations providing the following values of the remaining unknowns:

$$U_0(0) = k\frac{g_{-1}(L)p_1 - g_{-2}(L)p_2}{g_{-3}(L)g_{-1}(L) - [g_{-2}(L)]^2}$$
$$U_1(0) = k\frac{g_{-3}(L)p_2 - g_{-2}(L)p_1}{g_{-3}(L)\cdot g_{-1}(L) - [g_{-2}(L)]^2} \tag{27}$$

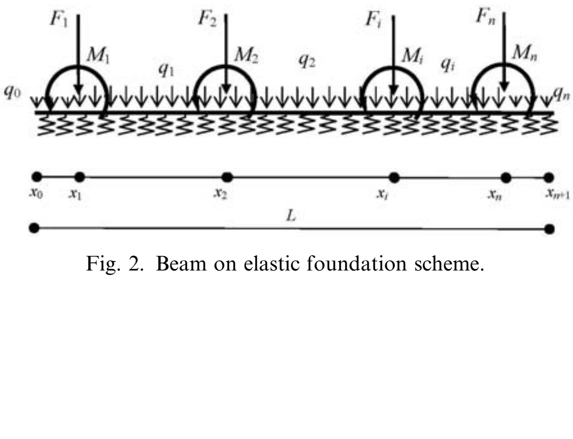


Fig. 2. Beam on elastic foundation scheme.

where

$$p_1 = \sum_{i=1}^{n} f_{i,-1}\bar{g}_{-4}(L - x_i) + \sum_{i=1}^{n} f_{i,-2}\bar{g}_{-1}(L - x_i) + \sum_{i=1}^{n} f_{i,-2}R_{-1}(L - x_i) + \sum_{j=0}^{n} q_j\Big(\bar{g}_{-3}(L - x_j) - \bar{g}_{-3}(L - x_j - a_j)\Big)$$

$$p_2 = \sum_{i=1}^{n} f_{i,-1}\bar{g}_{-3}(L - x_i) + \sum_{i=1}^{n} f_{i,-2}\bar{g}_{-4}(L - x_i) + \sum_{j=0}^{n} q_j\big(\bar{g}_{-2}(L - x_j) - \bar{g}_{-2}(L - x_j - a_j)\big) \quad (28)$$

By simple substitutions, the following expression is obtained for the response deflection:

$$\begin{aligned} u(x) &= k\frac{g_{-1}(L)p_1 - g_{-2}(L)p_2}{g_{-3}(L)g_{-1}(L) - [g_{-2}(L)]^2} g_{-4}(x) \\ &+ k\frac{g_{-3}(L)p_2 - g_{-2}(L)p_1}{g_{-3}(L)\cdot g_{-1}(L) - [g_{-2}(L)]^2} g_{-3}(x) + \\ &+ \sum_{i=1}^{n} \tilde{f}_{i,-1}\bar{g}_{-1}(x - x_i) + \sum_{i=1}^{n} \tilde{f}_{i,-2}\bar{g}_{-2}(x - x_i) \\ &+ \sum_{j=0}^{n} \tilde{q}_j\big(\bar{g}_0(x - x_j) - \bar{g}_0(x - x_j - a_j)\big) \end{aligned} \quad (29)$$

The corresponding slope, bending moment and shear force can be obtained by the simple derivations of Equation (29), according to the rules given in Equations (21).

Equation (29) represents the closed form solution for the case under consideration. It is important to note that the application of the classical approach does not allow one to find a solution of this kind, requiring the solution of a system of $4(n+1)$ equations.

## APPLICATIONS

In this section two illustrative numerical examples are developed in order to show the versatility and the simplicity of the formulation proposed in the previous sections. The following two structures are analyzed: (1) a clamped-clamped beam with an internal hinge and a bearing support, loaded by two forces, a moment, uniformly distributed, and linear distributed loads acting along the axis; (2) a realistic foundation beam.

The scheme of the first educational example is shown in Fig. 3. An elastic foundation modulus $k$ = 200 MN/m$^2$, a normal concrete Young modulus for beam $E$ = 30 GPa, and an inertial modulus for the beam cross section $I$ = 0.03333 m$^4$ have been chosen. The numerical values of the loads acting on the beam are reported in Table 1.

According to equations (19), (23) and (25) the solution reads:

$$\begin{aligned} u(x) &= \sum_{i=0}^{3} U_i(0)g_{-4+i}(x) + \big[\tilde{q}_1\big(\bar{g}_0(x - x_0) \\ &- \bar{g}_0(x - x_0 - a_1)\big) + \tilde{q}_2\bar{g}_0(x - x_3) + \\ &- \big(\tilde{q}_2 + \Delta\tilde{q}_2\big)\bar{g}_0(x - x_3 - a_2) + \frac{1}{a_2}\Delta\tilde{q}_2 \\ &\big(\bar{g}_1(x - x_3) - \bar{g}_1(x - x_3 - a_2)\big)\big] + \\ &+ \tilde{f}_{1,-1}\bar{g}_{-1}(x - x_1) + \tilde{f}_{2,-1}\bar{g}_{-1}(x - x_2) \\ &+ \tilde{f}_{3,-2}\bar{g}_{-2}(x - x_2) + \tilde{\rho}_{1,-3}\bar{g}_{-3}(x - x_1) \\ &+ \tilde{\rho}_{2,-1}\bar{g}_{-1}(x - x_3) \end{aligned} \quad (30)$$

In order to evaluate the integration constant, in addition to the four boundary condition at the ends of the beam, two other conditions have to be imposed, respectively at the internal hinge, $M(x_1) = -EI\,u''(x_1) = 0$, and at the bearing support, $u(x_3) = 0$. The conditions at the left end simply give: $U_0(0) = U_1(0) = 0$; while the remaining four conditions define a linear four equation system and provides the following four unknowns:

$$U''(0) = 5.231 \cdot 10^{-5}\, m^{-1},$$
$$U'''(0) = 8.327 \cdot 10^{-5}\, m^{-2},$$
$$\rho_1^{(-3)} = EI\Delta\widehat{\varphi}_1 = -907.462\, kN \cdot m^2,$$
$$\rho_2^{(-1)} = \widehat{F}_1 = -177.561\, kN.$$

It is important to remark that, if the classical approach is applied, the solution of a linear sixteen

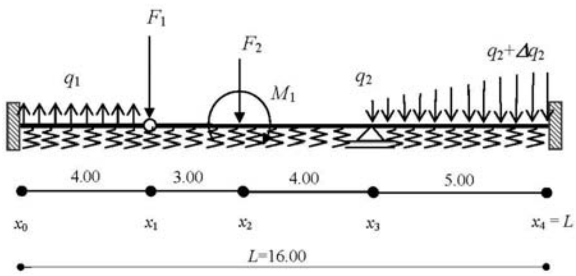


Fig. 3. Example 1: educational example scheme.

Table 1. Load values for example 1

| | |
|---|---|
| $x_1$ = 4.00 m | $F_1 = f_1^{(-1)}$ = 500 kN |
| $x_2$ = 7.00 m | $F_2 = f_2^{(-1)}$ = 300 kN; $M_1 = f_3^{(-2)}$ = 500 kN m |
| $x_0 \le x \le x_1$; $a_1 = x_1 - x_0$ = 4.00 m | $q_1$ = 20 kN/m, |
| $x_3 \le x \le x_4$; $a_2 = x_4 - x_3$ = 5.00 m | $q_2$ = 20 kN/m, $\Delta q_2$ = 15 kN/m |

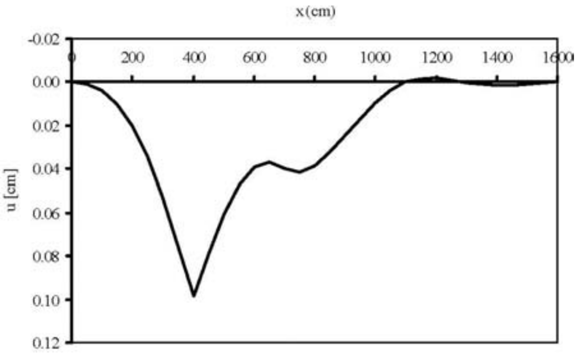


Fig. 4. Example 1: deflection diagram.

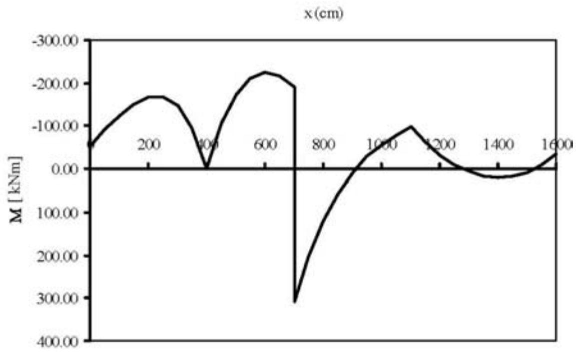


Fig. 6. Example 1: bending moment diagram.

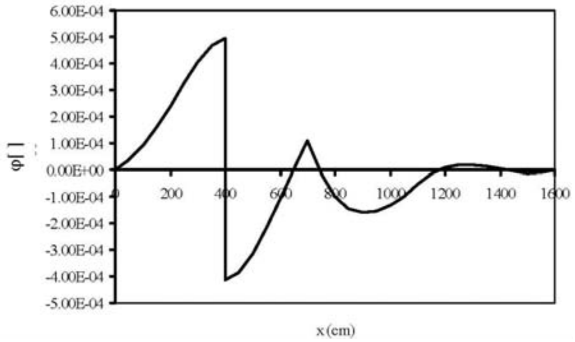


Fig. 5. Example 1: slope diagram.

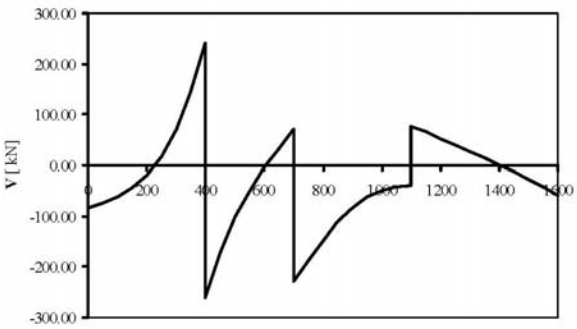


Fig. 7. Example 1: shear force diagram.

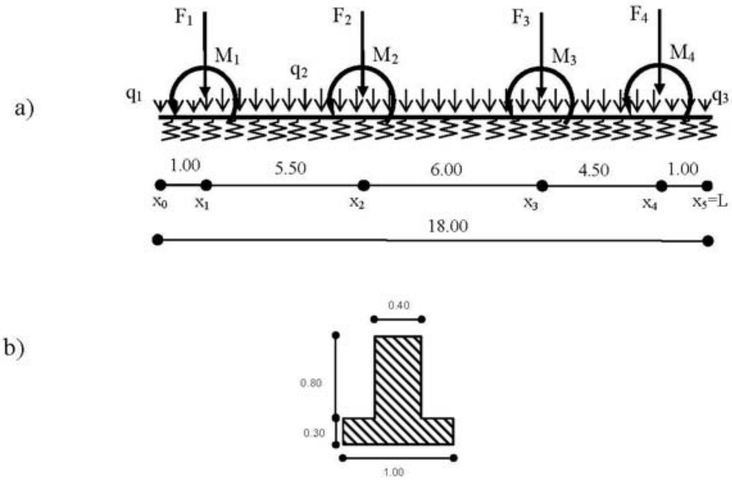


Fig. 8. Example 2: (a) beam scheme; (b) transversal section (lengths in meters).

Table 2. Load values for example 2

| | |
|---|---|
| $x_1$ = 1.00 m | $F_1$=$f_1^{(-1)}$= 500 kN, $M_1$=$f_2^{(-2)}$= 110 kN m |
| $x_2$ = 6.50 m | $F_2$=$f_3^{(-1)}$= 1000 kN, $M_2$=$f_4^{(-2)}$= 240 kN m |
| $x_3$ = 12.50 m | $F_3$=$f_5^{(-1)}$= 950 kN, $M_3$=$f_6^{(-2)}$= 200 kN m |
| $x_4$ = 17.00 m | $F_4$ =$f_7^{(-1)}$= 400 kN, $M_4$=$f_8^{(-2)}$= 90 kN m |
| $x_0 \le x \le x_{1;}$ $a_1 = x_1 - x_0$ = 1.00 m | $q_1$ = 15.5 kN/m |
| $x_1 \le x \le x_{2;}$ $a_2 = x_2 - x_1$ = 5.50 m | $q_2$ = 25.0 kN/m |
| $x_3 \le x \le x_{4;}$ $a_3 = x_3 - x_2$ = 6.00 m | $q_3$ = 25.0 kN/m |
| $x_4 \le x \le x_{5;}$ $a_4 = x_4 - x_3$ =4.50 m | $q_2$ = 25.0 kN/m |
| $x_5 \le x \le x_{6;}$ $a_5 = x_5 - x_4$ =1.00 m | $q_3$ = 15.5 kN/m |

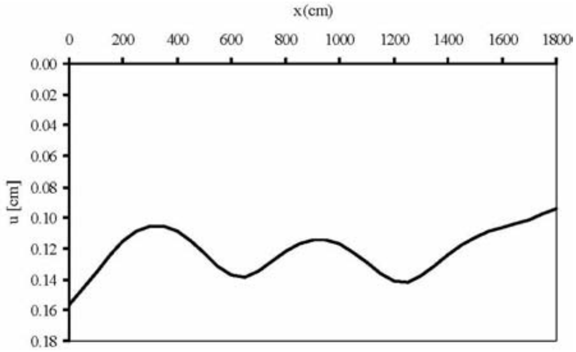


Fig. 9. Example 2: deflection diagram.

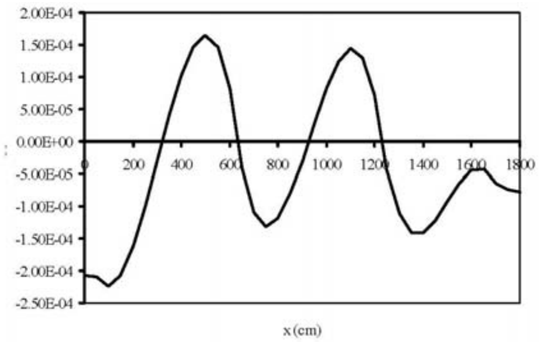


Fig. 10. Example 2: slope diagram.

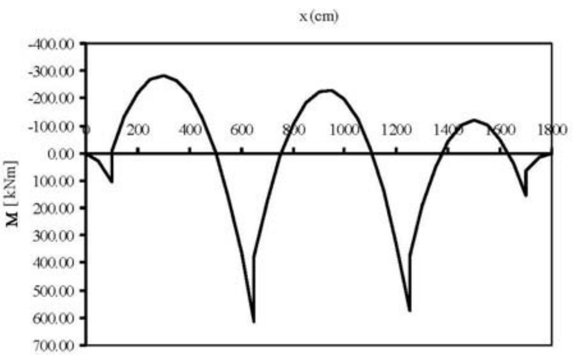


Fig. 11. Example 2: bending moment diagram.

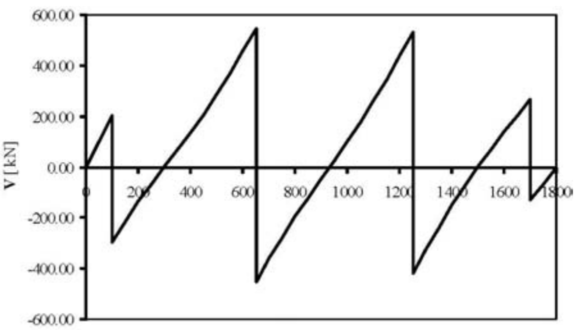


Fig. 12. Example 2: shear force diagram.

equation system had to be found. The deflection, slope, bending moment and shear force diagrams are plotted respectively in Figs 4–7.

In the second example a foundation beam of a three bay frame with dead self-load, axial forces and bending moments transmitted by the columns, is analyzed (Fig. 8). In this case, a value of elastic foundation modulus $k = 150$ MN/m$^2$, a normal concrete Young modulus for beam $E = 30$ GPa and a inertial modulus for beam cross section $I = 0.06615$ m$^4$ are adopted.

The numerical values of the loads acting on the beam are reported in Table 2.
By applying the proper substitutions, Equations (27) give the following values of the constants:

$$U_0(0) = k\frac{g_{-1}(L)p_1 - g_{-2}(L)p_2}{g_{-3}(L)g_{-1}(L) - [g_{-2}(L)]^2} = 1.562 \cdot 10^{-3}\ m$$

$$U_1(0) = k\frac{g_{-3}(L)p_2 - g_{-2}(L)p_1}{g_{-3}(L) \cdot g_{-1}(L) - [g_{-2}(L)]^2} = -2.072 \cdot 10^{-4}$$

The diagrams of deflection, slope, bending moment and shear force, obtained by derivation of Equation (29) through the derivation rules given in Equations (38), are shown respectively in Figs 9–12. In this case, if a classic approach were used, a twenty linear equation system would have to be solved.

## CONCLUSIONS

A new simplified formulation for the solution of beams on Winkler foundations with discontinuities due to loads and constraints has been presented. It is based on the use of the generalized functions and leads to reducing the number of boundary conditions to be imposed for finding the solution with respect to those necessary when the classical approach is used. In the simple case when discontinuities depend only on external loads, the proposed approach is able to give the exact closed form solution, in terms of deflection, slope, bending moment and shear force. The expressions of these solutions can be used directly, without solving any system of linear equations, as required by the application of the classical approach.

In any case, even when the closed form expression of the solution is not directly obtained, the proposed approach allows one to reduce drastically the number of boundary conditions to be imposed and, hence, the computational effort related to the problem solution.

Two illustrative examples highlighted the feasibility of the approach that can be easily applied in many fields of civil engineering where the solution to these problems is required, as, for example, in the case of the foundation analysis.

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