Int. J. Engng Ed. Vol. 25, No. 1, pp. 65–74, 2009
Printed in Great Britain.

0949-149X/91 $3.00+0.00

An Alternative Approach to Finding Beam Reactions and Deflections: Method of Model Formulas*

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This paper is intended to contribute an alternative approach—method of model formulas—to finding statically indeterminate reactions and deflections of elastic beams under loading. A set of four equations are first derived and then employed as model formulas. These formulas account for the flexural rigidity of the beam, concentrated loads, and linearly distributed loads. Thus, the proposed method of model formulas can effectively be applied to solve most beam problems involving reactions and deflections, encountered in the teaching of mechanics of materials and in engineering practice. A variety of examples are included in the paper.

Keywords: beam; reaction; slope; deflection; singularity function; model formulas

NOMENCLATURE

L	total length of beam *ab* to which model formulas are to be applied
EI	flexural rigidity of beam *ab*
q	loading function accounting for all loads acting on beam *ab*
V	shear force at position x of beam *ab*
V_a	shear force at left end a ($x = 0$) of beam *ab*
V_b	shear force at right end b ($x = L$) of beam *ab*
M	bending moment at position x of beam *ab*
M_a	bending moment at left end a ($x = 0$) of beam *ab*
M_b	bending moment at right end b ($x = L$) of beam *ab*
P	concentrated force at $x = x_P$
K	concentrated moment at $x = x_K$
w_0	beginning intensity of a distributed force at $x = x_w$
w_1	ending intensity of a distributed force at $x = u_w$
m_0	intensity of a uniformly distributed moment beginning at $x = x_m$ and ending at $x = u_m$
θ_a	slope of beam at its left end a ($x = 0$)
θ_b	slope of beam at its right end b ($x = L$)
y'	slope of beam at position x
y_a	deflection of beam at its left end a ($x = 0$)
y_b	deflection of beam at its right end b ($x = L$)
y	deflection of beam at position x
$< \ldots >$	angle brackets enclosing argument ... of a singularity function; cf., Equations (1)–(4)

* Accepted 14 May 2008.

INTRODUCTION

ALL BEAMS CONSIDERED in this paper are elastic beams, which are longitudinal members subjected to transverse loads. The major methods established for determining deflections of beams in mechanics of materials may include: (a) the method of double integration (*with* or *without* the use of singularity functions), (b) the method of superposition, (c) the method using moment–area theorems, (d) the method using Castigliano's theorem, (e) the conjugate beam method, and (f) the method of segments. These methods have been described in the literature and textbooks [1–12].

This paper significantly extends the main idea in the method of segments, as presented in [9], and does much to generalize earlier established formulas [4, 10] into general model formulas for studying beam reactions and deflections. The prerequisite to effective understanding and application of the alternative approach—the method of model formulas—proposed in this paper is a basic familiarity with the rudiments of singularity functions. Compared with the published method of segments [9], the proposed method has many advantages; e.g., there is a drastic reduction in the number of beam **segments** and resulting **simultaneous equations** involved in studying the beams *whenever* multiple concentrated loads or linearly distributed loads are found somewhere on the beam. The proposed method offers an independent and effective method for mechanics educators and practitioners when it comes to determining reactions and deflections of beams. Therefore, this paper contributes to the expansion of one's list of analytical tools and effective means of performing *independent assessment* or *checking* the solutions for beam problems that have been obtained by other methods [1–12].

In sharp contrast to the method of segments [9], which does not use singularity functions, the proposed method emerges as *superior* because one rarely needs to divide a beam into multiple segments for study and the method is not prone to generating inordinate numbers of simultaneous equations in the solution of beam problems, even if any of the following conditions exist:

- The beam carries multiple concentrated loads (forces or moments).
- The beam has one or more simple supports *not* at its ends.
- The beam has linearly distributed loads *not* starting at its left end.
- The beam has linearly distributed loads *not* ending at its right end.

For instance, if we fast forward to Example 1, given in this paper, for a moment, we see that the method of model formulas can treat the entire beam in this example as just one segment AB and can involve *only* the solution of *two* simultaneous equations in finding the values for the *two* unknowns: θ_A and y_A. However, if the method of segments as presented in [9] was employed, these *two* unknowns in the example would need to be solved in conjunction with the solving of another *ten* unknowns: θ_C, y_C, V_C, M_C, θ_D, y_D, V_D, M_D, V_B, and M_B as a package. In other words, the method of segments is much less efficient: it requires dividing the beam AB into three segments, AC, CD, and DB, to generate *twelve* simultaneous equations (six material equations *plus* six equilibrium equations) for solving the *twelve* unknowns before the values for θ_A and y_A could finally be found. In fact, if the other examples in this paper were to be solved with the method of segments [9], large sets of simultaneous equations would have to be generated and solved.

For the benefit of a wider readership who may have a variety of specialties in mechanics and to avoid or minimize any possible misunderstandings, this paper briefly goes over the adopted sign conventions and relevant singularity functions for beams. Readers, who are familiar with the rudiments of beams and singularity functions, may *skip* the next two section of this paper. The application of the model formulas is direct and requires no integration or writing of continuity equations. These model formulas can readily be extended to the analysis of beams that have discontinuity in slope (e.g., at hinge connections) or in flexural rigidity (e.g., in stepped beams) by dividing the beam into segments, where each segment has no such discontinuity, as demonstrated in Example 7. In the event of a *nonlinearly* distributed load acting on the beam, the model formulas in this paper can, of course, be modified by the user for a specific nonlinearly distributed load.

SIGN CONVENTIONS FOR BEAMS

The free-body diagram for a beam ab that has a constant flexural rigidity EI and carries selected typical loads is shown in Fig. 1. Generally, the sign conventions for shear forces, moments, and applied loads acting on a beam are as follows:

- A *shear force* is *positive* if it acts upward on the left (or downward on the right) face of the beam element (e.g., $\mathbf{V}_a$ at the left end a, and $\mathbf{V}_b$ at the right end b in Fig. 1).
- At the ends of the beam, a *moment* is *positive* if it tends to cause compression in the top fiber of the beam (e.g., $\mathbf{M}_a$ at the left end a, and $\mathbf{M}_b$ at the right end b in Fig. 1).
- If not at ends of the beam, a *moment* is *positive* if it tends to cause compression in the top fiber of

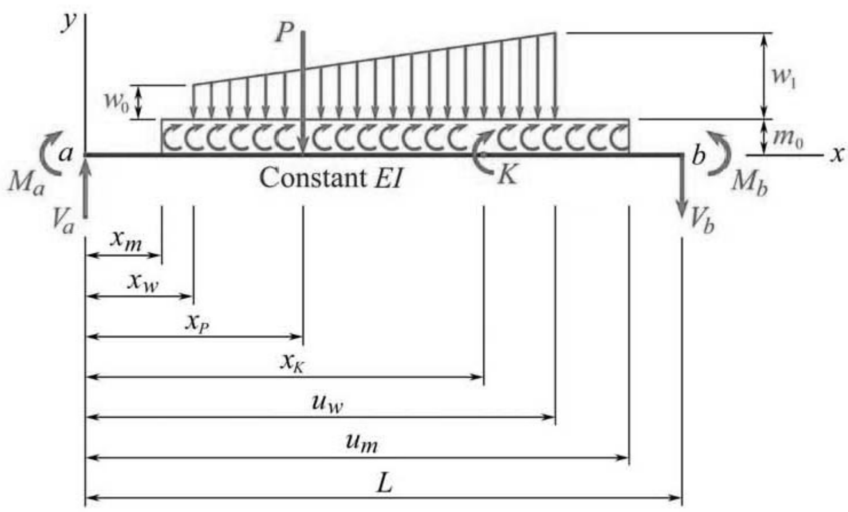

Fig. 1. Positive directions of shear forces, moments, and applied loads.

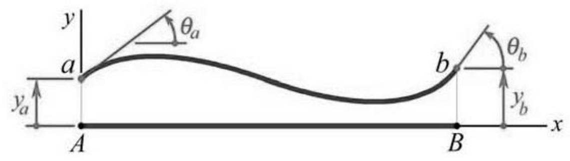

Fig. 2. Slopes and deflections of a beam displaced from AB to ab.

the beam just to the right of the position where it acts (e.g., the concentrated moment $\mathbf{K} = K \circlearrowleft$ and the uniformly distributed moment with intensity m_0 in Fig. 1).

- A *concentrated force* or a *distributed force* applied to the beam is *positive* if it is directed downward (e.g., the concentrated force $\mathbf{P} = P \downarrow$, the linearly distributed force with intensity w_0 on the left side and intensity w_1 on the right side in Fig. 1, where the distribution becomes uniform if $w_0 = w_1$).

As shown in Fig. 2, we adopt the following sign conventions for slope and deflection of a beam:

- A *positive slope* is a counterclockwise angular displacement (e.g., θ_a and θ_b in Fig. 2).
- A *positive deflection* is an upward linear displacement (e.g., y_a and y_b in Fig. 2).

SINGULARITY FUNCTIONS

As in most textbooks, the argument of a singularity function in this paper is shown enclosed by angle brackets (i.e., $< >$), while the argument of a regular function is enclosed by parentheses [i.e., ( )]. The rudiments of singularity functions [11, 12] are summarized as follows:

$$< x - a >^n = (x-a)^n \text{ if } x - a \geq 0 \text{ and } n > 0 \quad (1)$$

$$< x - a >^n = 1 \text{ if } x - a \geq 0 \text{ and } n = 0 \quad (2)$$

$$< x - a >^n = 0 \text{ if } x - a < 0 \quad (3)$$

$$< x - a >^n = 0 \text{ if } n < 0 \quad (4)$$

$$\int_{-\infty}^{x} < x - a >^n dx = \frac{1}{n+1} < x - a >^{n+1} \text{ if } n > 0 \quad (5)$$

$$\int_{-\infty}^{x} < x - a >^n dx = < x - a >^{n+1} \text{ if } n \leq 0 \quad (6)$$

$$\frac{d}{dx} < x - a >^n = n < x - a >^{n-1} \text{ if } n > 0 \quad (7)$$

$$\frac{d}{dx} < x - a >^n = < x - a >^{n-1} \text{ if } n \leq 0 \quad (8)$$

Equations (2) and (3) imply that, in using singularity functions for beams, we take

$$b^0 = 1 \text{ for } b \geq 0 \quad (9)$$

$$b^0 = 0 \text{ for } b < 0 \quad (10)$$

DERIVATION OF MODEL FORMULAS

Using singularity functions for beams [11, 12], we may write the loading function q, shear force V, and bending moment M for the beam ab in Fig. 1 as follows:

$$\begin{aligned} q = {} & V_a < x >^{-1} + M_a < x >^{-2} \\ & - P < x - x_P >^{-1} + K < x - x_K >^{-2} \\ & - w_0 < x - x_w >^0 \\ & - \frac{w_1 - w_0}{u_w - x_w} < x - x_w >^1 + w_1 < x - u_w >^0 \\ & + \frac{w_1 - w_0}{u_w - x_w} < x - u_w >^1 + m_0 < x - x_m >^{-1} \\ & - m_0 < x - u_m >^{-1} \end{aligned} \quad (11)$$

$$\begin{aligned} V = {} & V_a < x >^0 + M_a < x >^{-1} \\ & - P < x - x_P >^0 + K < x - x_K >^{-1} \\ & - w_0 < x - x_w >^1 \\ & - \frac{w_1 - w_0}{2(u_w - x_w)} < x - x_w >^2 + w_1 < x - u_w >^1 \\ & + \frac{w_1 - w_0}{2(u_w - x_w)} < x - u_w >^2 + m_0 < x - x_m >^0 \\ & - m_0 < x - u_m >^0 \end{aligned} \quad (12)$$

$$\begin{aligned} M = {} & V_a < x >^1 + M_a < x >^0 - P < x - x_P >^1 \\ & + K < x - x_K >^0 - \frac{w_0}{2} < x - x_w >^2 \\ & - \frac{w_1 - w_0}{6(u_w - x_w)} < x - x_w >^3 + \frac{w_1}{2} < x - u_w >^2 \\ & + \frac{w_1 - w_0}{6(u_w - x_w)} < x - u_w >^3 + m_0 < x - x_m >^1 \\ & - m_0 < x - u_m >^1 \end{aligned} \quad (13)$$

Letting the constant flexural rigidity of the beam ab be EI, y be the deflection, y' be the slope, and y'' be the second derivative of y with respect to the abscissa x, which defines the position of the section of the beam under consideration, we may apply the relation $EIy'' = M$ to write

$$EIy'' = V_a < x >^1 + M_a < x >^0 - P < x - x_P >^1 + K < x - x_K >^0 - \frac{w_0}{2} < x - x_w >^2 - \frac{w_1 - w_0}{6\,(u_w - x_w)} < x - x_w >^3 + \frac{w_1}{2} < x - u_w >^2 + \frac{w_1 - w_0}{6\,(u_w - x_w)} < x - u_w >^3 + m_0 < x - x_m >^1 - m_0 < x - u_m >^1 \quad (14)$$

$$EIy' = \frac{V_a}{2} < x >^2 + M_a < x >^1 - \frac{P}{2} < x - x_P >^2 + K < x - x_K >^1 - \frac{w_0}{6} < x - x_w >^3 - \frac{w_1 - w_0}{24(u_w - x_w)} < x - x_w >^4 + \frac{w_1}{6} < x - u_w >^3 + \frac{w_1 - w_0}{24(u_w - x_w)} < x - u_w >^4 + \frac{m_0}{2} < x - x_m >^2 - \frac{m_0}{2} < x - u_m >^2 + C_1 \quad (15)$$

$$EIy = \frac{V_a}{6} < x >^3 + \frac{M_a}{2} < x >^2 - \frac{P}{6} < x - x_P >^3 + \frac{K}{2} < x - x_K >^2 - \frac{w_0}{24} < x - x_w >^4 - \frac{w_1 - w_0}{120\,(u_w - x_w)} < x - x_w >^5 + \frac{w_1}{24} < x - u_w >^4 + \frac{w_1 - w_0}{120\,(u_w - x_w)} < x - u_w >^5 + \frac{m_0}{6} < x - x_m >^3 - \frac{m_0}{6} < x - u_m >^3 + C_1 x + C_2 \quad (16)$$

The slope and deflection of the beam in Fig. 1 at its left end a (i.e., at $x = 0$) are θ_a and y_a, respectively, as illustrated in Fig. 2. Imposition of these two boundary conditions on Equations (15) and (16) yields the values for the constants of integration C_1 and C_2 as follows:

$$C_1 = EI\theta_a \quad (17)$$

$$C_2 = EIy_a \quad (18)$$

Substituting Equations (17) and (18) into Equations (15) and (16), we obtain the *model formulas* for the slope y' and deflection y, at any position x of the beam ab in Fig. 1, as follows:

$$y' = \theta_a + \frac{V_a}{2EI} x^2 + \frac{M_a}{EI} x - \frac{P}{2EI} < x - x_P >^2 + \frac{K}{EI} < x - x_K >^1 - \frac{w_0}{6EI} < x - x_w >^3 - \frac{w_1 - w_0}{24EI\,(u_w - x_w)} < x - x_w >^4 + \frac{w_1}{6EI} < x - u_w >^3 + \frac{w_1 - w_0}{24EI\,(u_w - x_w)} < x - u_w >^4 + \frac{m_0}{2EI} < x - x_m >^2 - \frac{m_0}{2EI} < x - u_m >^2 \quad (19)$$

$$y = y_a + \theta_a x + \frac{V_a}{6EI} x^3 + \frac{M_a}{2EI} x^2 - \frac{P}{6EI} < x - x_P >^3 + \frac{K}{2EI} < x - x_K >^2 - \frac{w_0}{24EI} < x - x_w >^4 - \frac{w_1 - w_0}{120\,EI\,(u_w - x_w)} < x - x_w >^5 + \frac{w_1}{24\,EI} < x - u_w >^4 + \frac{w_1 - w_0}{120\,EI\,(u_w - x_w)} < x - u_w >^5 + \frac{m_0}{6\,EI} < x - x_m >^3 - \frac{m_0}{6\,EI} < x - u_m >^3 \quad (20)$$

By letting $x = L$ in Equations (19) and (20), we obtain the *model formulas* for the slope θ_b and deflection y_b at the right end b of the beam ab, as illustrated in Fig. 2, as follows:

$$\theta_b = \theta_a + \frac{V_a L^2}{2EI} + \frac{M_a L}{EI} - \frac{P}{2EI}(L - x_P)^2 + \frac{K}{EI}(L - x_K) - \frac{w_0}{6EI}(L - x_w)^3 - \frac{w_1 - w_0}{24EI\,(u_w - x_w)}(L - x_w)^4 + \frac{w_1}{6EI}(L - u_w)^3 + \frac{w_1 - w_0}{24EI\,(u_w - x_w)}(L - u_w)^4 + \frac{m_0}{2EI}(L - x_m)^2 - \frac{m_0}{2EI}(L - u_m)^2 \quad (21)$$

$$\begin{aligned}y_b = y_a &+ \theta_a L + \frac{V_a L^3}{6EI} + \frac{M_a L^2}{2EI} - \frac{P}{6EI}(L - x_P)^3 \\ &+ \frac{K}{2EI}(L - x_K)^2 - \frac{w_0}{24EI}(L - x_w)^4 \\ &- \frac{w_1 - w_0}{120EI(u_w - x_w)}(L - x_w)^5 + \frac{w_1}{24EI}(L - u_w)^4 \\ &+ \frac{w_1 - w_0}{120EI(u_w - x_w)}(L - u_w)^5 \\ &+ \frac{m_0}{6EI}(L - x_m)^3 - \frac{m_0}{6EI}(L - u_m)^3 \end{aligned} \quad (22)$$

APPLICATIONS OF MODEL FORMULAS

The preceding set of *four model formulas*, highlighted in Equations (19) through (22), forms the basis upon which an alternative approach—method of model formulas—is established for analyzing statically indeterminate reactions at supports, as well as the slopes and deflections, of beams. A beam may carry a variety of loads, as illustrated in Fig. 1, where each type of load may be repeated and accounted for accordingly.

Note that L in the model formulas in Equations (19) through (22) is a *parameter* representing the *total length* of the beam segment. In other words, this L is to be replaced by the *total length* of the beam segment, to which the model formulas are applied. The model formulas have already accounted for the boundary conditions of the beam at its ends. In particular, notice that this method allows one to treat reactions at interior supports (i.e., those *not* at the ends of the beam) as applied concentrated forces or moments, as appropriate. All one has to do is simply to impose the additional boundary conditions at the points of interior supports for the beam segment. Thus, statically indeterminate reactions as well as slopes and deflections of beams can be solved.

A beam needs to be divided into segments for analysis only if (a) it is a combined beam (e.g., a *Gerber beam*) having discontinuities in slope at hinge connections between segments, and (b) it contains segments with different flexural rigidities (e.g., a stepped beam). The *method of model formulas* proposed in this paper can best be understood with illustrations. Therefore, both simple and more challenging problems are included in the following examples.

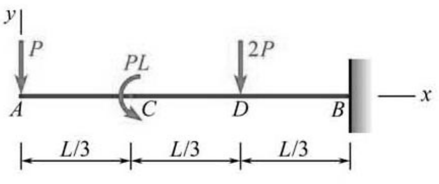

Fig. 3. Cantilever beam carrying two forces and a moment.

Example 1

A cantilever beam AB with constant flexural rigidity EI and length L is acted on by two concentrated forces of magnitudes P and $2P$ and a concentrated moment of magnitude PL as shown in Fig. 3. Determine the slope θ_A and deflection y_A at end A.

Solution

In applying the *method of model formulas*, we need to follow the sign conventions as illustrated in Figs. 1 and 2. At end A, the moment M_A is zero and the shear force V_A is $-P$. At end B, the slope θ_B and deflection y_B are both zero. Note in the model formulas that we have $x_K = L/3$, $K = -PL$, $x_P = 2L/3$, and the concentrated force at D is $2P$. Applying the model formulas in Equations (21) and (22), successively, to this beam as a single segment AB, we write

$$\begin{aligned}0 = \theta_A &+ \frac{-PL^2}{2EI} + 0 - \frac{2P}{2EI}\left(L - \frac{2L}{3}\right)^2 \\ &+ \frac{-PL}{EI}\left(L - \frac{L}{3}\right) - 0 - 0 + 0 + 0 + 0 - 0\end{aligned}$$

$$\begin{aligned}0 = y_A + \theta_A L &+ \frac{-PL^3}{6EI} + 0 - \frac{2P}{6EI}\left(L - \frac{2L}{3}\right)^3 \\ &+ \frac{-PL}{2EI}\left(L - \frac{L}{3}\right)^2 - 0 - 0 + 0 + 0 + 0 - 0\end{aligned}$$

The preceding *two* simultaneous equations yield

$$\theta_A = \frac{23\,PL^2}{18\,EI} \qquad y_A = -\frac{71\,PL^3}{81\,EI}$$

We report that

$$\theta_A = \frac{23\,PL^2}{18\,EI}\circlearrowleft \qquad y_A = \frac{71\,PL^3}{81\,EI}\downarrow$$

Example 2

A beam AB with constant flexural rigidity EI and length L, a fixed support at A, a roller support at B, and carrying a linearly distributed load is shown in Fig. 4. Determine (a) the vertical reaction force $\mathbf{A}_y$ and reaction moment $\mathbf{M}_A$ at A, (b) the slope θ_B at B.

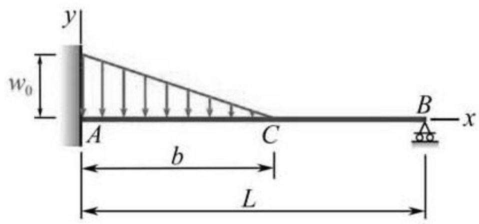

Fig. 4. Propped cantilever beam carrying linearly distributed load.

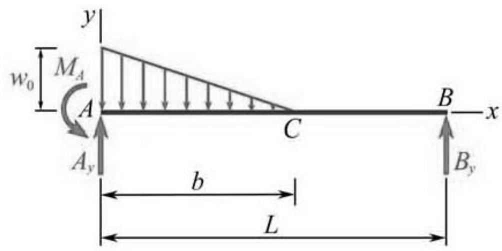

Fig. 5. Free-body diagram of the cantilever beam in Fig. 4.

Solution

From the free-body diagram shown in Fig. 5, we see that the beam under consideration is statically indeterminate to the *first* degree. Naturally, the *method of model formulas* can solve for both statically indeterminate reactions and deflections of beams.

The boundary conditions reveal that the slope θ_A and deflection y_A at A, as well as the deflection y_B at B, are all zero. We note in the model formulas that $x_w = 0$, $u_w = b$, and $w_1 = 0$. Applying the model formulas in Equations (21) and (22), successively, to the entire beam, we write

$$\theta_B = 0 + \frac{A_y L^2}{2EI} + \frac{-M_A L}{EI} - 0 + 0 - \frac{w_0}{6EI}L^3 - \frac{-w_0}{24EIb}L^4 + 0 + \frac{-w_0}{24EIb}(L-b)^4 + 0 - 0$$

$$0 = 0 + 0 + \frac{A_y L^3}{6EI} + \frac{-M_A L^2}{2EI} - 0 + 0 - \frac{w_0}{24EI}L^4 - \frac{-w_0}{120EIb}L^5 + 0 + \frac{-w_0}{120EIb}(L-b)^5 + 0 - 0$$

For equilibrium of the beam in Fig. 5, we set $+\circlearrowleft \Sigma M_B = 0$:

$$M_A - LA_y + \left(L - \frac{b}{3}\right)\left(\frac{w_0 b}{2}\right) = 0$$

The preceding *three* simultaneous equations yield:

$$A_y = \frac{w_0 b\,(20L^3 - 5b^2 L + b^3)}{40L^3}$$

$$M_A = \frac{w_0 b^2 (20L^2 - 15bL + 3b^2)}{120L^2}$$

$$\theta_B = \frac{w_0 b^3 (5L - 3b)}{240LEI}$$

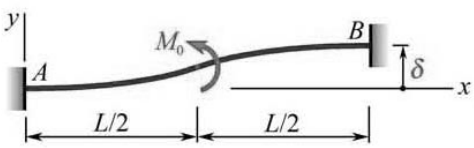

Fig. 6. Relative vertical shifting of supports in a loaded beam.

We report that

$$\mathbf{A}_y = \frac{w_0 b\,(20L^3 - 5b^2 L + b^3)}{40L^3} \uparrow$$

$$\mathbf{M}_A = \frac{w_0 b^2 (20L^2 - 15bL + 3b^2)}{120L^2} \circlearrowleft$$

$$\theta_B = \frac{w_0 b^3 (5L - 3b)}{240LEI} \circlearrowleft$$

Example 3

A fix-ended beam AB with constant flexural rigidity EI and length L is loaded with a concentrated moment $\mathbf{M}_0$ and its right end B is shifted upward by an amount δ, without rotation, as shown in Fig. 6. Determine (a) the vertical reaction force $\mathbf{A}_y$ and the reaction moment $\mathbf{M}_A$ at A, (b) the deflection y of the beam at any position x.

Solution

This beam is statically indeterminate to the *second* degree. At the fixed end A, the deflection y_A and slope θ_A are zero. At the fixed end B, the deflection y_B is δ, but the slope θ_B is zero. Applying the model formulas in Equations (21) and (22), successively, to this beam, we write

$$0 = 0 + \frac{A_y L^2}{2EI} + \frac{M_A L}{EI} - 0 + \frac{-M_0}{EI}\left(L - \frac{L}{2}\right) - 0 - 0 + 0 + 0 + 0 - 0$$

$$\delta = 0 + 0 + \frac{A_y L^3}{6EI} + \frac{M_A L^2}{2EI} - 0 + \frac{-M_0}{2EI}\left(L - \frac{L}{2}\right)^2 - 0 - 0 + 0 + 0 + 0 - 0$$

The preceding *two* simultaneous equations yield

$$A_y = \frac{3L^2 M_0 - 24EI\delta}{2L^3} \qquad M_A = \frac{24EI\delta - L^2 M_0}{4L^2}$$

We report that

$$\mathbf{A}_y = \frac{3L^2 M_0 - 24EI\delta}{2L^3} \uparrow$$

$$\mathbf{M}_A = \frac{24EI\delta - L^2 M_0}{4L^2} \circlearrowright$$

Substituting the obtained values of A_y and M_A into the model formula in Equation (20), we write

$$y = 0 + 0 + \frac{A_y}{6EI}x^3 + \frac{M_A}{2EI}x^2 - 0 + 0 - 0 - 0 + 0 + 0 + 0 - 0$$

$$y = \left(\frac{3\delta}{L^2} - \frac{M_0}{8EI}\right)x^2 - \left(\frac{2\delta}{L^3} - \frac{M_0}{4EIL}\right)x^3$$

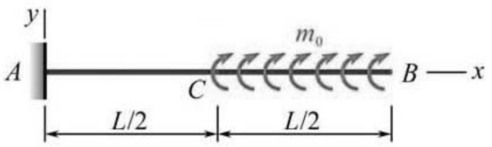

Fig. 7. Cantilever beam carrying a uniformly distributed moment.

Example 4

A cantilever beam AB with constant flexural rigidity EI and length L carries a uniformly distributed moment of intensity m_0 over half of its length L as shown in Fig. 7. Determine (a) the slope θ_B and deflection y_B at B, (b) the deflection y of the beam at any position x.

Solution

The free-body diagram of the beam AB, which is in equilibrium as shown in Fig. 8, indicates that the beam has only a counterclockwise reaction moment of magnitude $m_0L/2$ at its end A besides a uniformly distributed moment of intensity m_0 over half of its length L.

Figures 7 and 8 reveal that the deflection y_A, slope θ_A, and the shear force A_y at end A are all zero. At end B, the moment M_B and shear force B_y are both zero. Applying the model formulas in Equations (21) and (22), successively, to this beam and noting that $x_m = L/2$, we write

$$\theta_B = 0 + 0 + \frac{(-m_0L/2)L}{EI} - 0 + 0 - 0 - 0 + 0 + 0 + \frac{m_0}{2\,EI}\left(L - \frac{L}{2}\right)^2 - 0$$

$$y_B = 0 + 0 + 0 + \frac{(-m_0L/2)\,L^2}{2\,EI} - 0 + 0 - 0 - 0 + 0 + 0 + \frac{m_0}{6\,EI}\left(L - \frac{L}{2}\right)^3 - 0$$

The preceding *two* simultaneous equations yield

$$\theta_B = -\frac{3\,m_0\,L^2}{8\,EI} \qquad y_B = -\frac{11\,m_0\,L^3}{48\,EI}$$

We report that

$$\theta_B = \frac{3\,m_0\,L^2}{8\,EI} \circlearrowright \qquad y_B = \frac{11\,m_0\,L^3}{48\,EI} \downarrow$$

Substituting the obtained values of θ_B and y_B into the model formula in Equation (20), we write

$$y = 0 + 0 + 0 + \frac{-m_0L/2}{2\,EI}x^2 - 0 + 0 - 0 - 0 + 0 + 0 + \frac{m_0}{6\,EI}\left(x - \frac{L}{2}\right)^3 - 0$$

$$y = -\frac{m_0}{48\,EI}\left[12\,L\,x^2 + (2x - L)^3\right]$$

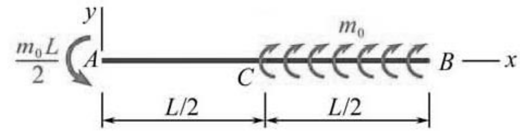

Fig. 8. Free-body diagram of the cantilever beam in Fig. 7.

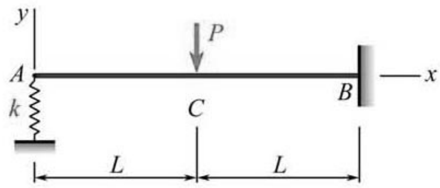

Fig. 9. Cantilever beam propped by a linear spring and carrying a concentrated force.

Example 5

A cantilever beam AB with constant flexural rigidity EI and length $2L$ is propped by a linear spring of modulus k, and it carries a concentrated force $\mathbf{P}$ at its midpoint C as shown in Fig. 9. Determine the slope θ_A and deflection y_A at A.

Solution

At end A of this beam, the moment M_A is zero and the shear force $V_A = -ky_A$, which is based on the initial assumption that y_A is upward and the linear spring force of ky_A acts downward at end A. At end B, the slope θ_B and deflection y_B are both zero. Note that we need to replace the *parameter L* in the model formulas in Equations (21) and (22) with $2L$ for this beam AB. Letting $x_P = L$ and applying the model formulas in Equations (21) and (22), successively, to this beam, we write

$$0 = \theta_A + \frac{(-ky_A)\,(2L)^2}{2EI} + 0 - \frac{P}{2EI}(2L - L)^2 + 0 - 0 - 0 + 0 + 0 + 0 - 0$$

$$0 = y_A + \theta_A(2L) + \frac{(-ky_A)(2L)^3}{6EI} + 0 - \frac{P}{6EI}(2L - L)^3 + 0 - 0 - 0 + 0 + 0 + 0 - 0$$

The preceding *two* simultaneous equations yield

$$\theta_A = \frac{PL^2\,(3EI - 2\,kL^3)}{2\,EI(3EI + 8\,kL^3)}$$

$$y_A = -\frac{5PL^3}{2\,(3EI + 8\,kL^3)}$$

We report that

$$\theta_A = \frac{PL^2\,(3EI - 2\,kL^3)}{2\,EI(3EI + 8\,kL^3)} \circlearrowleft$$

$$y_A = \frac{5PL^3}{2\,(3EI + 8\,kL^3)} \downarrow$$

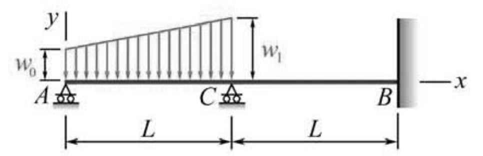

Fig. 10. Continuous beam carrying linearly distributed load.

Example 6

A continuous beam AB with constant flexural rigidity EI and total length $2L$ has a roller support at A, a roller support at C, a fixed support at B and carries a linearly distributed load as shown in Fig. 10. Determine (a) the vertical reaction force $\mathbf{A}_y$ and the slope θ_A at A, (b) the vertical reaction force $\mathbf{C}_y$ and the slope θ_C at C.

Solution

We note that this beam is statically indeterminate to the *second* degree, which may naturally be solved by the *method of model formulas*. We can simply treat the vertical reaction force $\mathbf{C}_y$ at C as an unknown applied concentrated force, directed upward, and notice that the beam AB has a total length of $2L$, which is to be used as the value for the *parameter* L in the model formulas in Equations (19) through (22). The boundary conditions of this beam reveal that the moment M_A and deflection y_A at A are zero, the slope θ_B and deflection y_B at B are zero, and the deflection y_C at C is zero. The shear force at the left end A is the vertical reaction force $\mathbf{A}_y$ at A, which may be assumed to be acting upward. Applying the model formulas in Equations (21) and (22) to the entire beam and using the model formula in Equation (20) to impose that $y_C = 0$ at C, in that *order*, we write

$$0 = \theta_A + \frac{A_y\,(2L)^2}{2\,EI} + 0 - \frac{-\,C_y}{2\,EI}(2L-L)^2 + 0$$
$$- \frac{w_0}{6\,EI}(2L)^3 - \frac{w_1 - w_0}{24\,EIL}(2L)^4$$
$$+ \frac{w_1}{6\,EI}(2L-L)^3 + \frac{w_1 - w_0}{24\,EIL}(2L-L)^4 + 0 - 0$$

$$0 = 0 + \theta_A(2L) + \frac{A_y(2L)^3}{6\,EI} + 0$$
$$- \frac{-\,C_y}{6\,EI}(2L-L)^3 + 0 - \frac{w_0}{24\,EI}(2L)^4$$
$$- \frac{w_1 - w_0}{120\,EIL}(2L)^5$$
$$+ \frac{w_1}{24\,EI}(2L-L)^4 + \frac{w_1 - w_0}{120\,EIL}(2L-L)^5 + 0 - 0$$

$$0 = 0 + \theta_A L + \frac{A_y}{6\,EI}L^3 + 0 - 0 + 0$$
$$- \frac{w_0}{24\,EI}L^4 - \frac{w_1 - w_0}{120\,EIL}L^5 + 0 + 0 + 0 - 0$$

The preceding *three* simultaneous equations yield

$$A_y = \frac{(21w_0 + 9w_1)L}{70} \qquad \theta_A = -\frac{(14w_0 + 11w_1)L^3}{840\,EI}$$
$$C_y = \frac{(7w_0 + 12\,w_1)L}{28}$$

We report that

$$\mathbf{A}_y = \frac{(21w_0 + 9w_1)L}{70} \uparrow$$
$$\theta_A = \frac{(14w_0 + 11w_1)L^3}{840\,EI} \circlearrowright$$
$$\mathbf{C}_y = \frac{(7w_0 + 12\,w_1)L}{28} \uparrow$$

The slope θ_C is simply y' evaluated at C, which is located at $x = L$. Applying the model formula in Equation (19) and utilizing the preceding solutions for θ_A and A_y, we write

$$\theta_C = \theta_A + \frac{A_y}{2\,EI}L^2 + 0 - 0 + 0 - \frac{w_0}{6\,EI}L^3$$
$$- \frac{w_1 - w_0}{24\,EI\,L}L^4 + 0 + 0 + 0 - 0$$
$$= \frac{(7w_0 + 8w_1)L^3}{840\,EI}$$

We report that

$$\theta_C = \frac{(7w_0 + 8w_1)L^3}{840\,EI} \circlearrowleft$$

Example 7

A stepped beam ABC carries a uniformly distributed load w_0 as shown in Fig. 11, where the segments AB and BC have flexural rigidities EI_1 and EI_2, respectively. Determine (a) the slopes θ_A, θ_B, and θ_C at A, B, and C, respectively, (b) the deflection y_B at B.

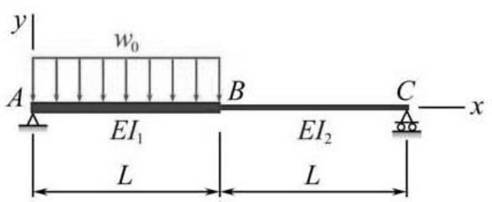

Fig. 11. Stepped beam carrying a uniformly distributed load.

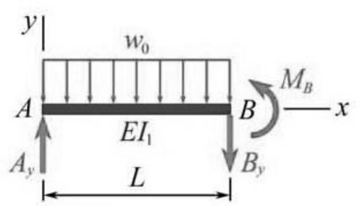

Fig. 12. Free-body diagram for segment *AB*.

Solution
Because of the *discontinuity in flexural rigidity* at B, this beam needs to be divided into two segments AB and BC for analysis in the solution. The boundary conditions of this beam reveal that the deflection y_A at A and the deflection y_C at C are zero.

Applying the model formulas in Equations (21) and (22), successively, to segment AB, as shown in Fig. 12, we write

$$\theta_B = \theta_A + \frac{A_y L^2}{2EI_1} + 0 - 0 + 0 - \frac{w_0}{6EI_1} L^3 - 0 + 0 + 0 + 0 - 0 \qquad (a)$$

$$y_B = 0 + \theta_A L_1 + \frac{A_y L^3}{6EI_1} + 0 - 0 + 0 - \frac{w_0}{24EI_1} L^4 - 0 + 0 + 0 + 0 - 0 \qquad (b)$$

For equilibrium of segment AB in Fig. 12, we write

$$+\uparrow \Sigma F_y = 0 : A_y - B_y - w_0 L = 0 \qquad (c)$$

$$+\circlearrowleft \Sigma M_B = 0 : -LA_y + \frac{w_0 L^2}{2} + M_B = 0 \qquad (d)$$

Applying the model formulas in Equations (21) and (22), successively, to segment BC, as shown in Fig. 13, we write

$$\theta_C = \theta_B + \frac{B_y L^2}{2EI_2} + \frac{M_B L}{EI_2} - 0 + 0 - 0 - 0 + 0 + 0 + 0 - 0 \qquad (e)$$

$$0 = y_B + \theta_B L + \frac{B_y L^3}{6EI_2} + \frac{M_B L^2}{2EI_2} - 0 + 0 - 0 - 0 + 0 + 0 + 0 - 0 \qquad (f)$$

For equilibrium of segment BC in Fig. 13, we write

$$+\uparrow \Sigma F_y = 0 : B_y - C_y = 0 \qquad (g)$$

$$+\circlearrowleft \Sigma M_C = 0 : -M_B - LB_y = 0 \qquad (h)$$

The preceding *eight* simultaneous equations yield

$$A_y = \frac{3w_0 L}{4} \qquad B_y = -\frac{w_0 L}{4}$$

$$C_y = -\frac{w_0 L}{4} \qquad M_B = \frac{w_0 L^2}{4}$$

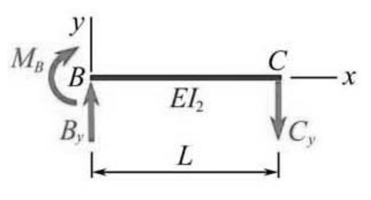

Fig. 13. Free-body diagram for segment BC.

$$\theta_A = -\frac{w_0 L^3(2I_1 + 7I_2)}{48EI_1I_2} \qquad \theta_B = -\frac{w_0 L^3(2I_1 - 3I_2)}{48EI_1I_2}$$

$$\theta_C = \frac{w_0 L^3(4I_1 + 3I_2)}{48EI_1I_2} \qquad y_B = -\frac{w_0 L^4(2I_1 + 3I_2)}{48EI_1I_2}$$

CONCLUSIONS

This paper is presented to share with educators and practitioners in mechanics a proposed general methodology that employs a set of four model formulas in solving problems involving statically indeterminate reactions at supports, as well as the slopes and deflections, of beams. These formulas, derived using singularity functions, provide the material equations, besides the equations of static equilibrium, for the solution of the problem. They are expressed in terms of the following: (a) flexural rigidity of the beam; (b) slopes and deflections, as well as shear forces and bending moments, at both ends of the beam; and (c) applied loads on the beam. Selected typical applied loads are illustrated in Fig. 1, which shows inclusion of a concentrated force and a concentrated moment, somewhere on the beam; a linearly distributed force over a portion of the beam; and a uniformly distributed moment over a portion of the beam.

The *method of model formulas* contains three major salient features: (i) eliminating in many problems the need to 'segment' the beam and drastically reduce the need to solve simultaneous equations, such as those in Examples 1 through 6; (ii) allowing the boundary conditions at certain supports to be readily imposed using also the model formulas, such as those in Examples 3 and 5; (iii) allowing one to treat unknown reactions at supports not at the ends of a beam simply as concentrated forces or moments, such as that in Example 6. These are salient features not matched by the original method of segments [9]. A beam needs to be divided into two or more segments for analysis only when it has discontinuity in slope or in flexural rigidity, such as that in Example 7. Nevertheless, one needs to remember that the *parameter L* in the model formulas represents the total length of the beam segment, to which the formulas are to be applied.

Seven carefully selected examples have been included to cover the gamut of possible questions and to illustrate the power and generality of the method. The rudiments of singularity functions are usually explained in undergraduate textbooks [11, 12] for sophomore or junior students who usually take a course in mechanics of materials and for senior students who usually take a course in mechanical or structural design in their undergraduate curricula. It is recommended that the method of model formulas be taught to students as an alternative approach, after first teaching them one or more of the traditionally established methods [1–12]. Thus, the method of model formulas may directly benefit and enrich the learning experience and learning outcome of upper class engineering students, as well as practising civil and mechanical engineers. Furthermore, the method of model formulas may readily serve as an independent and effective means to quickly assess or check the solutions obtained using other established methods.

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