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Bending Problem of Euler–Bernoulli Discontinuous Beams*

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The bending problem of Euler–Bernoulli discontinuous beams is a classic topic in mechanics. In this paper stepped beams with internal springs are addressed based on the theory of generalized functions. It is shown that in this context a closed-form expression may be given to the Green's functions due to point forces and, based on these, to the beam response to arbitrary loads, for any set of boundary conditions. The proposed solution method may be presented in a regular course in Mechanics of Solids and Strength of Materials for undergraduate students. It does not require an advanced knowledge of the theory of generalized functions but the knowledge of only a few basic concepts, most of which are generally presented in other courses such as, for instance, Dynamics of Structures. It is hoped that it may help students to address in a simple and effective way the many engineering applications involving discontinuous beams.

Keywords: static Green's functions; Euler–Bernoulli beam theory; discontinuous beams; flexural-stiffness steps; internal springs

NOMENCLATURE

$\mathbf{A}$ = coefficient matrix in set of boundary conditions
a, b = left and right end of interval spanned by loading function $p(x)$
a_{jk} = elements of coefficient matrix $\mathbf{A}$
$\mathbf{c}$ = vector of integration constants in beam differential equations
c_j = components of vector $\mathbf{c}$
$\mathbf{G}$ = matrix of functions for force Green's functions
$\mathbf{g}$ = vector of Green's functions
$\mathbf{g}^{(P)}$ = vector of functions for force Green's functions
g_u, g_φ, g_M, g_S = components of vector $\mathbf{g}$
g_{uj}, $g_{\varphi j}$ = functions in matrix $\mathbf{G}$
$g_u^{(P)}$, $g_\varphi^{(P)}$ = components of vector $\mathbf{g}^{(P)}$
EI = reference flexural stiffness of first uniform beam segment
EI_i = flexural stiffness of ith uniform beam segment between x_i and x_{i+1}.
$H(x-x_0)$ = unit-step function at $x = x_0$
I_j = closed-form integral
L = beam length
$M(x)$ = bending moment
N = number of discontinuities
P = point force in the z-direction
$p(x)$ = piecewise continuous transverse load
$\bar{p}(x)$ = loading function for load $p(x)$
$q^{[j]}(x)$ = jth-order primitive in rotation and deflection solution to beam differential equations
$R_n(x-x_0)$ = nth-order ramp function at $x = x_0$
$\mathbf{r}(x)$ = response vector to arbitrary loads
$S(x)$ = shear force
s = integration variable
$u(x)$ = deflection
V = reaction of roller support
$\mathbf{v}$ = vector in set of boundary conditions
v_j = components of vector $\mathbf{v}$
x, z = coordinate of reference system
x_i = discontinuity location
x_V = roller support location
x_i^{inf}, x_i^{sup} = right and left integration bounds in integral solution for arbitrary loads
y = location of applied point force
β_i = parameter for flexural-stiffness step at x_i
γ_i = parameter giving flexural stiffness of ith uniform beam segment in terms of reference flexural stiffness EI
$\delta(x-x_0)$ = Dirac delta at $x = x_0$
$\Pi\left(\frac{x-x_0}{\tau}\right)$ = unit-area rectangle function at $x = x_0$ spanning an interval τ
$\varphi(x)$ = rotation
$\chi(x, y)$ = function in integral solution for arbitrary loads
ψ = flexibility of internal springs or external elastic supports
$\Im(x, y, x_i)$ = polynomial function in integral solution for arbitrary loads
$\mathbb{N}$ = sets of indexes i corresponding to discontinuity locations x_i

Subscripts and superscripts
$[j]$ = jth-order primitive of a function
M = bending moment
S = shear force

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u = deflection
V = roller support
Δ = deflection discontinuity
Θ = rotation discontinuity
φ = rotation

INTRODUCTION

THE BASIC CONCEPTS of the Euler–Bernoulli (EB) beam bending problem are generally introduced to the students by reference to a uniform beam under continuous loads, the case in which the solution is defined over the whole beam length and is given in terms of four integration constants, to be computed by enforcing the boundary conditions (BC). In many applications, however, beams are discontinuous. This happens whenever the loads do not span the whole beam length, or discontinuities must be accounted for in the response variables such as: shear-force and bending-moment discontinuities due to external point loads or external supports; deflection and rotation discontinuities, either inelastic or due to internal springs; curvature discontinuities due to abrupt changes in the material properties and/or the cross-section geometry, in the so-called stepped beams. In particular, internal springs and stepwise reductions of flexural stiffness are frequent in engineering applications, to model the effects of damage [1, 2]. Stepped beams are also frequently used in bridge engineering to reduce weight and optimize strength [3, 4].

In regular courses in mechanics, the solution to the bending problem of discontinuous beams is generally constructed based on classical functions, by splitting the beam into N+1 uniform segments (if N is the total number of discontinuity locations) and integrating the equilibrium fourth-order differential equation for each segment. As a result, 4(N+1) unknown constants are to be computed by enforcing appropriate internal compatibility and continuity conditions at the discontinuity locations, along with the BC. This approach then leads to a numerical solution achieved by solving a system of 4(N+1) coupled algebraic equations.

The classical solution based on a beam decomposition into uniform segments is not the only solution referred to in textbooks of mechanics. In the literature, in fact, there has been a considerable effort to develop alternative and more efficient solutions. In general, they have been sought based on the theory of generalized functions. In this context Macaulay [5–7] was the first to show that a point force may be taken to be continuous by a pertinent generalized function, i.e. the well-known Dirac delta. In this way, by using simple integration rules for generalized functions, he showed that a solution to uniform beams under point forces may be built by enforcing the BC only, thus eliminating the need to enforce internal conditions at the location of the applied force [7]. An interesting review on Macaulay's method may be found in [8], where the author has also discussed its generalization to beams with along-axis constraints and presented a simple and effective methodology to introduce undergraduate students to the use of generalized functions for discontinuous beams.

Solutions based on generalized functions have been developed not only by Macaulay but also by Brungraber [9], Kanwal [10], Yavari *et al.* [11], Biondi and Caddemi [12], Failla and Santini [13]. They showed that a discontinuity in any response variable may be reverted to a pertinent generalized loading function in the beam equilibrium fourth-order differential equation. To a different extent, however, solutions in [9–13] all involve computing a number of unknowns by enforcing some internal conditions along with the BC, or the BC only [12]. That is, they are still numerical solutions. Also, some of these methods involve quite lengthy derivations that may be hardly illustrated to undergraduate students in regular courses in mechanics.

The aim of this paper is to show that a few basic concepts of the theory of generalized functions may be used, in a simple yet effective manner, to build closed-form solutions for discontinuous beams. The paper focuses specifically on stepped beams with internal springs acted upon by arbitrary loads. It will be shown that, based on simple rules of integration for generalized functions, the response variables may all be derived as closed-form functions of the discontinuity parameters, for any set of BCs. For this the Green's functions due to a point force will be first derived.

The paper develops as follows. After introducing the basic notation in the next section, the Green's functions are derived and the response to arbitrary loads is found; the material is presented with the aim of providing teachers and students with tools that are ready to use for applications. In the last sections the potential impact of the proposed method on teaching and learning is discussed and examples are given. A brief summary of the concepts of the theory of generalized functions required in the paper is outlined in the Appendix.

PROBLEM STATEMENT AND NOTATION

Consider the stepped beam with internal springs in Fig. 1. Be L the length and x_i the discontinuity locations, $0 < .... < x_{i-1} < x_i < ... < L$, for $i \in \mathbb{N}$, $\mathbb{N} = \{i : i = 1, 2, \ldots, \mathrm{N}\}$, to which $N + 1$ segments correspond over the beam length. Denote by EI_1 the flexural stiffness of the first segment, taken as reference flexural stiffness, i.e., $EI_1 = EI$, and by $EI_i = EI(1 - \gamma_i)$ the flexural stiffness of the i-th segment, where $\gamma_i < 1$ for $i = 2, \ldots, \mathrm{N} + 1$. Let $\beta_1 = \gamma_2/(1 - \gamma_2)$ and $\beta_i = \gamma_{i+1}/(1 - \gamma_{i+1}) - \gamma_i/(1 - \gamma_i)$, for $i \in \mathbb{N}$. That is, $\beta_i = 0$ if the flexural stiffness does not vary through $x = x_i$; $\beta_i < 0$ if the flexural stiffness increases through $x = x_i$ while $\beta_i > 0$ if it decreases. Further, let ψ_i^{Θ} and ψ_i^{Δ} be the

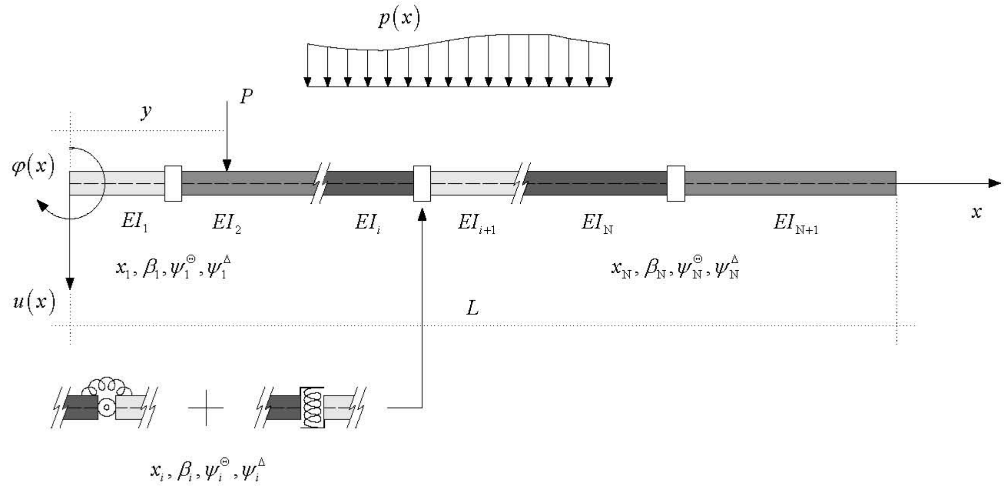

Fig. 1. Stepped beam with internal springs under static loads; the BC are arbitrary.

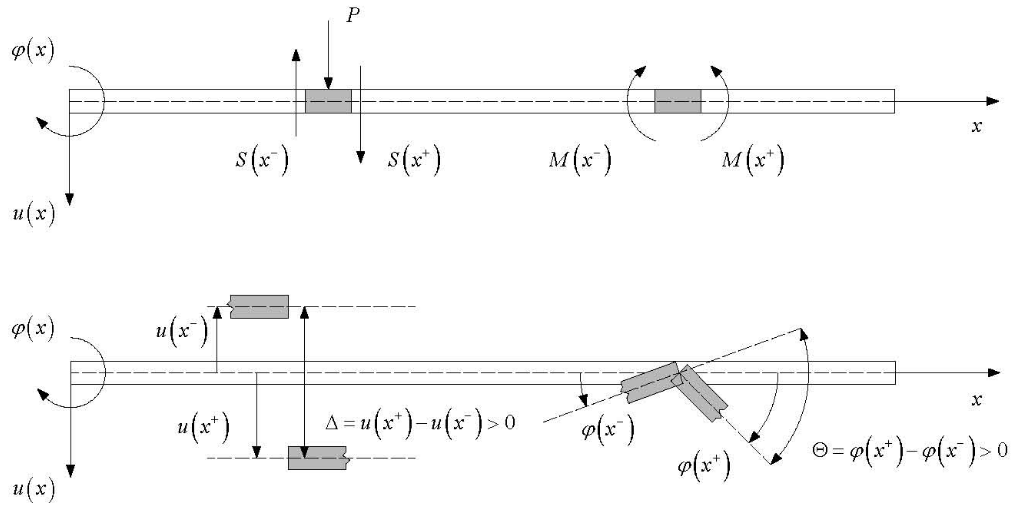

Fig. 2. Sign conventions for response variables of a discontinuous beam.

flexibilities of the internal rotational and translational springs. For generality, each discontinuity location is represented in Fig. 1 as a multi-discontinuity location where a flexural-stiffness step and internal springs occur simultaneously. However any discontinuity pattern may be built at a location x_i by proper selection of the discontinuity parameters: for instance, at a location x_i where only a rotational spring occurs $\beta_i = 0$ and $\psi_i^{\Delta} = 0$ shall be obviously set.

Let $u(x)$ be the deflection, $\varphi(x)$ the rotation, $M(x)$ the bending-moment and $S(x)$ the shear-force response variables. Positive sign conventions are set in Fig. 2.

STATIC GREEN'S FUNCTIONS OF DISCONTINUOUS BEAMS

Consider the stepped beam with internal springs shown in Fig. 1, acted on only by a point force P at $x = y,\ 0 < y < L$. For the consistency of subsequent developments, it shall be assumed that no deflection discontinuity occurs at $x = y$; a curvature and a rotation discontinuity are both allowed, however, at $x = y$.

According to the sign convention in Fig. 2, the differential equations governing the response variables may be written as

$$\frac{\bar{d}S(x)}{dx} = -P\delta(x - y) \tag{1}$$

$$\frac{\bar{d}M(x)}{dx} = S(x) \tag{2}$$

$$\frac{\bar{d}\varphi(x)}{dx} = -\frac{M(x)}{EI}\left[1 + \sum_{i\in\mathbb{N}} \beta_i H(x - x_i)\right] - \sum_{i\in\mathbb{N}} M(x_i)\psi_i^{\Theta}\delta(x - x_i) \tag{3}$$

$$\frac{\bar{d}u(x)}{dx} = \varphi(x) + \sum_{i\in\mathbb{N}} S(x_i)\psi_i^{\Delta}\delta(x - x_i) \quad (4)$$

where $\varphi(x_i^+) - \varphi(x_i^-) = -M(x_i)\psi_i^{\Theta}$ and $u(x_i^+) - u(x_i^-) = S(x_i)\psi_i^{\Delta}$, for $i \in \mathbb{N}$.

As explained in [8–13], the Dirac delta $P\delta(x - y)$ in Equation (1) results from the point force applied at $x = y$. Similarly, in Equation (3) the unit-step functions $M(x)/EI \cdot \beta_i H(x - x_i)$ and the Dirac deltas $M(x_i)\psi_i^{\Theta}\delta(x - x_i)$ reflect, respectively, the curvature discontinuities due to the flexural-stiffness steps and the rotation discontinuities due to the rotational springs at $x = x_i$, for $i \in \mathbb{N}$. Further, in Equation (4) the Dirac deltas $S(x_i)\psi_i^{\Delta}\delta(x - x_i)$ reflect the deflection discontinuities due to the translational springs at $x = x_i$, for $i \in \mathbb{N}$. All the derivatives in the l.h.s. of Equation (1) through Equation (4) are generalized derivatives (denoted by the bar over the differentiation symbol), due to the generalized functions in the corresponding r.h.s.

Equation (1) through Equation (4) may be integrated based on the integration rules reported in the Appendix. It yields

$$S(x) = -H(x - y) + c_1 \quad (5)$$

$$M(x) = -R_1(x - y) + c_1 x + c_2 \quad (6)$$

$$\varphi(x) = -\frac{q^{[1]}(x)}{EI} - \sum_{i\in\mathbb{N}} M(x_i)\psi_i^{\Theta} H(x - x_i) + c_3 \quad (7)$$

$$u(x) = -\frac{q^{[2]}(x)}{EI} - \sum_{i\in\mathbb{N}} M(x_i)\psi_i^{\Theta} R_1(x - x_i) + \sum_{i\in\mathbb{N}} S(x_i)\psi_i^{\Delta} H(x - x_i) + c_3 x + c_4 \quad (8)$$

where the c_j's are integration constants. Also, $q^{[1]}(x)$ and $q^{[2]}(x)$ are the first- and second-order primitives:

$$q^{[1]}(x) = M^{[1]}(x) + \sum_{i\in\mathbb{N}} \beta_i H(x - x_i)\left[M^{[1]}(x) - M^{[1]}(x_i)\right] \quad (9)$$

$$q^{[2]}(x) = M^{[2]}(x) + \sum_{i\in\mathbb{N}} \beta_i H(x - x_i)\left[M^{[2]}(x) - M^{[2]}(x_i) - M^{[1]}(x_i)R_1(x - x_i)\right] \quad (10)$$

where $M^{[1]}(x) = -R_2(x - y) + c_1 x^2/2 + c_2 x$ and $M^{[2]}(x) = -R_3(x - y) + c_1 x^3/6 + c_2 x^2/2$, that is, respectively, a first and second-order primitive of the bending-moment function (6). Note that Equation (5) is derived from Equation (A.7) and Equation (6) from Equation (A.11) for n=1. Equation (7) and Equation (8) are derived by applying Equation (A.14) for $f(x) = M(x)$ and $f(x) = M^{[1]}(x)$, respectively.

For $P = 1$, the response variables in Equation (5) through (8) represent the so-called influence coefficients of the discontinuous beam, also referred to in the literature as Force Green's Functions (FGFs). Here they will be denoted (with the obvious meaning of the subscripts) as

$$\mathbf{g}(x, y) = [\,g_u(x, y) \quad g_{\varphi}(x, y) \quad g_M(x, y) \quad g_S(x, y)\,]^T \quad (11)$$

where explicit dependence on the location of the applied force, $x = y$, is introduced. To derive $\mathbf{g}(x, y)$, replace Equation (9) and Equation (10) for $q^{[j]}(x)$ and perform simple manipulations to single out the terms that multiply each integration constant c_j and P in Equation (5) through Equation (8); then, if P=1 is set, the following general form is obtained

$$\mathbf{g}(x, y) = \mathbf{G}(x)\mathbf{c}(y) + \mathbf{g}^{(\mathrm{P})}(x, y) \quad (12)$$

where $\mathbf{c} = [\,c_1 \quad c_2 \quad c_3 \quad c_4\,]^T$ and

$$\mathbf{G} = \begin{bmatrix} g_{u1}(x) & g_{u2}(x) & x & 1 \\ g_{\varphi 1}(x) & g_{\varphi 2}(x) & 1 & 0 \\ x & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \quad (13)$$

$$\mathbf{g}^{(\mathrm{P})} = \begin{bmatrix} g_u^{(\mathrm{P})}(x, y) \\ g_{\varphi}^{(\mathrm{P})}(x, y) \\ -R_1(x - y) \\ -H(x - y) \end{bmatrix} \quad (14)$$

Elements of the matrix $\mathbf{G}$, of the vectors $\mathbf{g}^{(\mathrm{P})}$ and $\mathbf{c}$ are given in the following as functions of the discontinuity parameters; $\mathbf{c}$ depends, as is obvious, on the BC.

Elements of matrix $\mathbf{G}$

For the displacement $u(x)$, functions $g_{u1}(x)$ and $g_{u2}(x)$ in matrix $\mathbf{G}(x)$ are given as:

$$g_{u1}(x) = -\frac{1}{6EI}\left[x^3 + \sum_{i\in\mathbb{N}} 2\beta_i(x + 2x_i)R_2(x - x_i)\right] - \sum_{i\in\mathbb{N}} \psi_i^{\Theta} x_i R_1(x - x_i) + \sum_{i\in\mathbb{N}} \psi_i^{\Delta} H(x - x_i) \quad (15)$$

$$g_{u2}(x) = -\frac{1}{2EI}\left[x^2 + \sum_{i\in\mathbb{N}} 2\beta_i R_2(x - x_i)\right] - \sum_{i\in\mathbb{N}} \psi_i^{\Theta} R_1(x - x_i) \quad (16)$$

Further, for the rotation $\varphi(x)$ functions $g_{\varphi 1}(x)$ and $g_{\varphi 2}(x)$ are given as:

$$g_{\varphi 1}(x) = -\frac{1}{2EI}\left[x^2 + \sum_{i\in\mathbb{N}} \beta_i (x + x_i) R_1(x - x_i)\right] - \sum_{i\in\mathbb{N}} \psi_i^{\Theta} x_i H(x - x_i) \quad (17)$$

$$g_{\varphi 2}(x) = -\frac{1}{EI}\left[x + \sum_{i\in\mathbb{N}} \beta_i R_1(x - x_i)\right] - \sum_{i\in\mathbb{N}} \psi_i^{\Theta} H(x - x_i) \quad (18)$$

Elements of vector $\mathbf{g}^{(P)}(x, y)$

For the displacement $u(x)$, function $g_u^{(P)}(x, y)$ in vector $\mathbf{g}^{(P)}(x, y)$ is given as:

$$g_u^{(P)}(x, y) = \frac{R_3(x-y)}{EI} + \frac{1}{6EI}\sum_{i\in\mathbb{N}} \beta_i H(x - x_i) \left[6R_3(x-y) + 2(y + 2x_i - 3x)R_2(x_i - y)\right] + \sum_{i\in\mathbb{N}} \psi_i^{\Theta} R_1(x - x_i) R_1(x_i - y) - \sum_{i\in\mathbb{N}^I} \psi_i^{\Delta} H(x - x_i) H(x_i - y) \quad (19)$$

Also, for the rotation $\varphi(x)$ function $g_\varphi^{(P)}(x, y)$ is given as:

$$g_\varphi^{(P)}(x, y) = \frac{R_2(x-y)}{EI} + \frac{1}{EI}\sum_{i\in\mathbb{N}} \beta_i H(x - x_i) \left[R_2(x-y) - R_2(x_i - y)\right] + \sum_{i\in\mathbb{N}} \psi_i^{\Theta} R_1(x_i - y) H(x - x_i) \quad (20)$$

Vector of integration constants $\mathbf{c}$

First consider a clamped–clamped (CC) beam. Enforcing the BC leads to the matrix equation:

$$\mathbf{A}\mathbf{c} = \mathbf{v} \quad (21)$$

where $\mathbf{A}$ and $\mathbf{v}$ are a 4×4 matrix and a 4×1 vector given as:

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ a_{31} & a_{32} & L & 1 \\ a_{41} & a_{42} & 1 & 0 \end{bmatrix} \quad \mathbf{v} = \begin{bmatrix} 0 \\ 0 \\ v_3 \\ v_4 \end{bmatrix} \quad (22), (23)$$

The elements of $\mathbf{A}$ and $\mathbf{v}$ are listed below

$$a_{31} = -\frac{1}{6EI}\left[L^3 + \sum_{i\in\mathbb{N}} \beta_i (L - x_i)^2 (L + 2x_i)\right] - \sum_{i\in\mathbb{N}} \psi_i^{\Theta} x_i (L - x_i) + \sum_{i\in\mathbb{N}} \psi_i^{\Delta} \quad (24)$$

$$a_{32} = -\frac{1}{2EI}\left[L^2 + \sum_{i\in\mathbb{N}} \beta_i (L - x_i)^2\right] - \sum_{i\in\mathbb{N}} \psi_i^{\Theta} (L - x_i) \quad (25)$$

$$a_{41} = -\frac{1}{2EI}\left[L^2 + \sum_{i\in\mathbb{N}} \beta_i \left(L^2 - x_i^2\right)\right] - \sum_{i\in\mathbb{N}} \psi_i^{\Theta} x_i \quad (26)$$

$$a_{42} = -\frac{1}{EI}\left[L + \sum_{i\in\mathbb{N}} \beta_i (L - x_i)\right] - \sum_{i\in\mathbb{N}} \psi_i^{\Theta} \quad (27)$$

$$v_3(y) = \frac{1}{6EI}\left\{-(L-y)^3 + \sum_{i\in\mathbb{N}} \beta_i \left[-(L-y)^3 + 2(3L - y - 2x_i)R_2(x_i - y)\right]\right\} + - \sum_{i\in\mathbb{N}} \psi_i^{\Theta} (L - x_i) R_1(x_i - y) + \sum_{i\in\mathbb{N}^I} \psi_i^{\Delta} H(x_i - y) \quad (28)$$

$$v_4(y) = -\frac{1}{2EI}\left\{(L-y)^2 + \sum_{i\in\mathbb{N}} \beta_i \left[(L-y)^2 - 2R_2(x_i - y)\right]\right\} - \sum_{i\in\mathbb{N}} \psi_i^{\Theta} R_1(x_i - y) \quad (29)$$

The matrix $\mathbf{A}$ is invertible and solutions for $\mathbf{c}$ are given as:

$$c_1 = \frac{a_{32}v_4 - a_{42}v_3}{a_{32}a_{41} - a_{31}a_{42}}$$

$$c_2 = \frac{a_{41}v_3 - a_{31}v_4}{a_{32}a_{41} - a_{31}a_{42}}$$

$$c_3 = c_4 = 0 \quad (30)$$

The same reasoning can be followed to derive the vector of integration constants $\mathbf{c}$ for any set of BC. For most recurrent BC, the closed-form expressions of $\mathbf{c}$ are summarized in Table 1, where a_{31}, a_{32}, a_{41}, a_{42}, $v_3(y)$ and $v_4(y)$ are still given by Equations (24) through (29).

Based on Equations (15) through (29) it may be seen that the FGFs (12) yielding the deflection response are symmetric, i.e., $g_u(x, y) = g_u(y, x)$, as long as no deflection discontinuity occurs at x. It is also worth noting that the FGFs (12) also apply to a beam with end elastic restraints. For instance, for a clamped-clamped (CC) beam it may be verified that the response variables obtained for non-homogeneous BCs due to end elastic restraints of flexibilities

Table 1. Vector of integration constants **c** in terms of the beam discontinuity parameters. SS—simply-supported; CP—clamped-pinned; CSR—clamped-shear released; PSR—pinned-shear released; CF—clamped-free

BC	c_1	c_2	c_3	c_4	v_3	v_4
SS	$\frac{v_4}{L}$	0	$\frac{v_3 L - a_{31} v_4}{L^2}$	0	Equation (28)	$L - y$
CP	$\frac{a_{32} v_4 - v_3}{a_{32} L - a_{31}}$	$\frac{v_3 L - a_{31} v_4}{a_{32} L - a_{31}}$	0	0	Equation (28)	$L - y$
CSR	v_3	$\frac{v_4 - a_{41}}{a_{42}}$	0	0	1	Equation (29)
PSR	v_3	0	$v_4 - a_{41}$	0	1	Equation (29)
CF	v_3	$-v_3 L + v_4$	0	0	1	$L - y$

ψ_O^Δ, ψ_L^Δ, ψ_O^Θ and ψ_L^Θ are identical to the FGFs (12), when the latter are computed for homogeneous BCs (i.e., $u(0) = \varphi(0) = u(L) = \varphi(L) = 0$ for a CC beam) and internal springs located at $x = 0^+$ and $x = L^-$, with flexibilities $\psi_1^\Delta = \psi_O^\Delta$, $\psi_1^\Theta = \psi_O^\Theta$, $\psi_N^\Delta = \psi_L^\Delta$ and $\psi_N^\Theta = \psi_L^\Theta$.

The FGFs (12) will be now used to build the beam response to arbitrary loads.

SOLUTIONS FOR DISCONTINUOUS BEAMS UNDER STATIC LOADS

For generality, consider a piecewise-continuous loading function

$$p(x) = \bar{p}(x)[H(x-a) - H(x-b)]$$

for

$$0 \le a < b \le L \tag{31}$$

where $p(x)$ is positive if downward (consistently with sign conventions in Fig. 2). The symbol $\bar{p}(x)$ in Equation (31) denotes an arbitrary continuous function for a which a fourth-order primitive is assumed to exist, as generally encountered in engineering applications [11, 12].

If $p(x)$ is applied to the beam in Fig. 1, based on the FGFs (12) the response variables can be written as

$$\mathbf{r}(x) = [u(x)\varphi(x)M(x)S(x)]^T = \\ = \int_0^L \mathbf{g}(x,y)p(y)dy = \mathbf{G}(x)\int_a^b \mathbf{c}(y)\bar{p}(y)dy \\ + \int_a^b \mathbf{g}^{(\mathrm{P})}(x,y)\bar{p}(y)dy \tag{32}$$

Solutions to the integrals in Equation (32) may be found based on the integration rules discussed in the Appendix, along with standard rules of integration by parts for classical functions.

To elucidate this concept, consider first the second integral in the r.h.s. of Equation (32). In view of $\mathbf{g}^{(\mathrm{P})}(x,y)$ given by Equation (14), it involves integrals in the two general forms

$$\text{(i)} \quad \int_a^b R_n(x-y)\bar{p}(y)dy, \text{ for } n = 0, 1, ..., 3 \tag{33}$$

$$\text{(ii)} \quad \int_a^b \Im(x,y,x_i)H(x_i - y)\bar{p}(y)dy = \\ \int_{x_i^{\mathrm{inf}}}^{x_i^{\mathrm{sup}}} \Im(x,y,x_i)\bar{p}(y)dy \tag{34}$$

where $\Im(x,y,x_i)$ is a polynomial function of x, y and x_i of the third order at most, $x_i^{\mathrm{inf}} = \min\{x_i, a\}$ and $x_i^{\mathrm{sup}} = \min\{x_i, b\}$, for $i \in \mathbb{N}$ (i.e., if $x_i \le a < b$, the integral vanishes). On the other hand, in view of $\mathbf{c}(y)$ given by Equation (30) and in Table 1, the first integral in the r.h.s. of Equation (32) involves only integrals in the form (ii).

Solutions to integrals in the form (i)–(ii) are easy to find, as explained as follows.

Solution to integrals in form (i)

Equation (33) may be rewritten as

$$\int_a^b R_n(x-y)\bar{p}(y)dy = \int_a^b \chi(x,y)H(x-y)dy \tag{35}$$

where $\chi(x,y) = \frac{(x-y)^n}{n!}\bar{p}(y)$. Based on Equation (A.15) it yields

$$\int_a^b \chi(x,y)H(x-y)dy \\ = \left\{\left[\chi^{[1]}(x,y) - \chi^{[1]}(x,x)\right]H(x-y)\right\}_a^b \tag{36}$$

In Equation (36) the symbol $\chi^{[1]}(x,y)$ denotes the first order primitive

$$\chi^{[1]}(x,y) = \int \frac{(x-y)^n}{n!}\bar{p}(y)dy \\ = \frac{1}{n!}\sum_{j=1}^{n+1}(-1)^{j-1}\bar{p}^{[j]}(y)\frac{d^{j-1}(x-y)^n}{dy^{j-1}} \tag{37}$$

where $\bar{p}^{[j]}(y)$ denotes the jth order primitive of $\bar{p}(y)$ in Equation (31). Note that Equation (37) is obtained by integration by parts for classical functions, where $\bar{p}(y)$ is repetitively integrated and $(x-y)^n$ is repeatedly differentiated until the last derivative vanishes.

Solutions to integrals in form (ii)

For Equation (34) the same rules of integration by parts used in Equation (37) obviously apply, being differentiable the polynomial functions $\Im(x, y, x_i)$. Formulas are not repeated for brevity.

EDUCATIONAL DATA

The proposed method should be presented in a two-hour lecture. Further, one-to-two review hours are suggested as needed to discuss a few applications. The basic concepts of the continuous beam bending problem are assumed to be known.

The first hour of lecture should be limited to introducing only the few concepts of generalized calculus summarized in the Appendix. The basic concepts of Dirac delta, unit-step and ramp functions should first be presented. Since they are also part of regular courses such as Structural Dynamics (e.g., to model impulsive and ramp loads, see for instance [14]), in this context emphasis should also be placed on the interdisciplinary use of generalized functions and their importance in mechanics. Then, the generalized integral (A.13) and corresponding derivative (A.14) should be introduced. Note that both Equations may easily be explained based on the definitions of Dirac delta, unit-step function and the derivative of a product according to classical calculus. It appears therefore that the proposed method does not require an advanced knowledge of the theory of generalized calculus and this is deemed important for a positive impact on teaching and learning.

The second hour of the lecture should develop in the following steps:

1. As a first step, the differential equations governing the response variables, Equations (1) through (4), should be written down. This can be done based on the concepts of generalized functions illustrated in the first hour of the lecture. In this context attention should be drawn to the fact that the generalized differential Equations (1) through (4) account inherently for those compatibility and continuity conditions that, instead, should be individually set at each discontinuity location when building a classical solution based on a beam decomposition into uniform segments. This is a considerable simplification involved by using generalized functions and should be presented to the students.
2. As a second step, the integration rules illustrated in the first hour of the lecture should be recalled to explain how to derive the general form (12) for the FGFs. Equations (13) and (14) may then be given to the students as formulas ready to use for applications. In this context the expressions for the integration constants c_j should be also given. That is, Equations (30) for a CC beam and the results in Table 1 for various BCs.
3. As a third step, the criteria to build the beam response to arbitrary loads should be explained. This may be done based on the generalized integral (A.15) and standard rules of integration by parts for classical functions. Study cases should subsequently be developed by the students during the review hours.

The students could run the applications using symbolic software packages such as Mathematica or Matlab, which allow a straightforward implementation of the formulas given above. There are several potential applications of the proposed method that could be suggested to the students. The FGFs (12), in fact, are the influence coefficients of discontinuous beams. As explained in classic mechanics texts, they may be then be used to build solutions for statically redundant structures; to study the effects of moving loads on bridges; and to formulate the eigen problem of distributed-parameters structures [15].

For any of these applications, the students should be encouraged to perform a sensitivity analysis on the results; that is, they should build and compare various solutions for different discontinuity parameters. In the authors' opinion, this is an important step in gaining a better physical insight into the mechanical behavior of discontinuous beams, for which the formulas given in this paper appear particularly suitable, since they can be readily updated for any change in the discontinuity pattern (flexibilities of the internal springs, amplitudes of the flexural-stiffness step, location of the discontinuities along the beam axis that may also change their relative position). In this context it is also worth remarking that, if this task was pursued by building a classical numerical solution based on a beam domain decomposition, the students should update the set of solving equations as soon as the discontinuities pattern changes, at the expense of rather tedious manipulations, which may hardly be carried out during a lecture.

Finally, further potential applications of the proposed method may concern optimal design. As sensitivity analysis, in fact, optimal design requires building a relevant number of solutions for changing beam parameters and, in this case, a closed-form solution is highly desirable compared with the numerical solutions that are available to date.

EXAMPLES

Consider the cantilever beam in Fig. 3. A flexural-stiffness step occurs at $x = x_1$ and a rotational

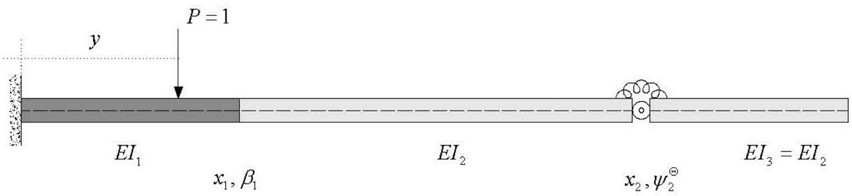

Fig. 3(a). Stepped cantilever beam with a rotational spring under a unit point force.

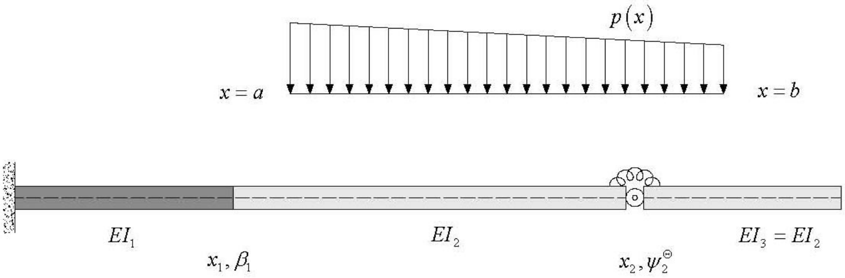

Fig. 3(b). Stepped cantilever beam with a rotational spring under a trapezoidal load.

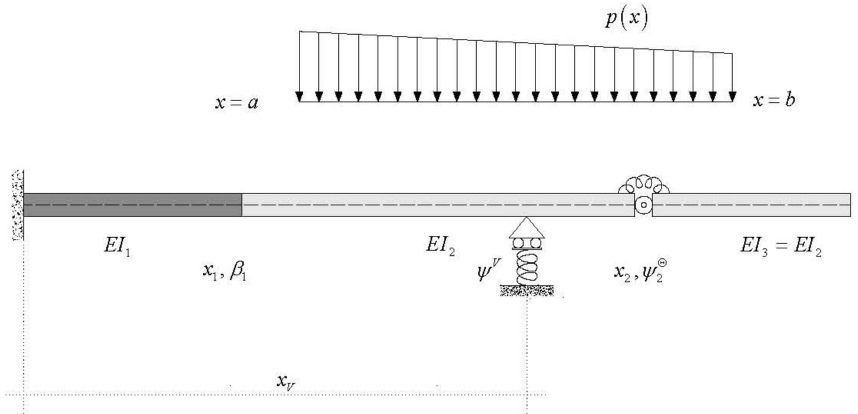

Fig. 3(c). Stepped cantilever beam with a rotational spring and a roller support, under a trapezoidal load.

spring at $x = x_2$. In Fig. 3 $x_1 < x_2$, but it may be also $x_1 \geq x_2$, since the formulas in the sections above hold for any discontinuity pattern. Assume that the deflection response is sought for the following study cases: (i) beam subjected to a point force $P = 1$ at an arbitrary location $x = y$ (Fig. 3(a)); (ii) beam subjected to the trapezoidal load $\bar{p}(x) = \bar{p}_0(2L - x)/L$, $\bar{p}_0 = \text{cost}$, distributed over an arbitrary interval [a, b] (Fig. 3(b)); (iii) beam subjected to the trapezoidal load in Fig. 3(c) and supported by a roller of flexibility ψ^V at an arbitrary location $x = x_V$.

The study cases (i), (ii) and (iii) may all be solved based on the FGFs (12), where the vector of integration constants **c** pertinent to a cantilever beam are reported in Table 1.

Based on Equation (12) and $c_1(y) = 1.0$, $c_2(y) = -y$, $c_3 = c_4 = 0$ (see Table 1), the deflection response to a point load $P = 1$ is given by

$$\begin{aligned} g_u(x,y) &= g_{u1}(x)c_1(y) + g_{u2}(x)c_2(y) \\ &\quad + xc_3(y) + c_4(y) + g_u^{(\mathrm{P})}(x,y) \\ &= g_{u1}(x) - g_{u2}(x)y + g_u^{(\mathrm{P})}(x,y) \end{aligned} \tag{38}$$

where

$$\begin{aligned} g_{u1}(x) = -\frac{1}{6EI}\left[x^3 + 2\beta_1(x + 2x_1)R_2(x - x_1)\right] \\ - \psi_2^{\Theta} x_2 R_1(x - x_2) \end{aligned} \tag{39}$$

$$g_{u2}(x) = -\frac{1}{2EI}\left[x^2 + 2\beta_1 R_2(x - x_1)\right] - \psi_2^{\Theta} R_1(x - x_2) \quad (40)$$

and

$$g_u^{(\mathrm{P})}(x,y) = \frac{R_3(x-y)}{EI} + \frac{\beta_1 H(x-x_1)}{6EI}[6R_3(x-y) + 2(y + 2x_1 - 3x)R_2(x_1 - y)] + \psi_2^{\Theta} R_1(x-x_2)R_1(x_2-y) \quad (41)$$

Study case (ii)

In this case, the deflection response is given as

$$u(x) = \int_a^b g_u(x,y)\bar{p}(y)dy = g_{u1}(x)\int_a^b \bar{p}(y)dy - g_{u2}(x)\int_a^b y\bar{p}(y)dy + \int_a^b g_u^{(\mathrm{P})}(x,y)\bar{p}(y)dy \quad (42)$$

The first two integrals in Equation (42) are trivial. It is also seen that

$$\int_a^b g_u^{(\mathrm{P})}(x,y)\bar{p}(y)dy = I_1(x) + I_2(x) + I_3(x) \quad (43)$$

where I_1, I_2 and I_3 are the closed-form integrals given as:

- $$I_1(x) = \frac{1+\beta_1 H(x-x_1)}{EI}\int_a^b R_3(x-y)\bar{p}(y)dy = \left[-\chi^{[1]}(x,x) + \chi^{[1]}(x,b)\right]H(x-b) + \left[\chi^{[1]}(x,x) - \chi^{[1]}(x,a)\right]H(x-a) \quad (44)$$

where

$$\chi^{[1]}(x,y) = \int \frac{(x-y)^3}{3!}\bar{p}(y)dy = \frac{1}{3!}\sum_{j=1}^{4}(-1)^{j-1}\bar{p}^{[j]}(y)\frac{d^{j-1}(x-y)^3}{dy^{j-1}} \quad (45)$$

for

$$\bar{p}^{[j]}(y) = \frac{\bar{p}_0}{2L}(-1)^j\frac{(2L-y)^{j+1}}{(j+1)!} \qquad j = 1,2... \quad (46)$$

Also,

- $$I_2(x) = \frac{\beta_1 H(x-x_1)}{6EI}\int_{x_1^{\inf}}^{x_1^{\sup}} \Im(x,y,x_1)\bar{p}(y)dy = \sum_{j=1}^{4}\left[(-1)^{j-1}\bar{p}^{[j]}(y)\frac{d^{j-1}\Im}{dy^{j-1}}\right]_{x_1^{\inf}}^{x_1^{\sup}} \quad (47)$$

where $\Im(x,y,x_1) = (y + 2x_1 - 3x)(x_1 - y)^2$ and, therefore,

$$\frac{d\Im}{dy} = 3(y-x_1)(y+x_1-2x)$$
$$\frac{d^2\Im}{dy^2} = 6(y-x) \quad (48, 49)$$

Finally,

- $$I_3(x) = \psi_2^{\Theta} R_1(x-x_2)\int_{x_2^{\inf}}^{x_2^{\sup}}(x_2-y)\bar{p}(y)dy = \sum_{j=1}^{2}\left[(-1)^{j-1}\bar{p}^{[j]}(y)\frac{d^{j-1}(x_2-y)}{dy^{j-1}}\right]_{x_2^{\inf}}^{x_2^{\sup}} \quad (50)$$

It is interesting to remark that, if $x_2 \le a < b$, the integral I_3 vanishes. That is, in this case the quote part of deflection response due to the spring located at $x = x_2$ is given only by the first two integrals in the r.h.s. of Equation (42), which yield

$$\psi_2^{\Theta} R_1(x-x_2)\int_a^b (y-x_2)\bar{p}(y)dy \quad (51)$$

Recognize in Equation (51) the deflection response (zero for $x \le x_2$, linearly-varying for $x > x_2$) due to the rotation of the spring located at $x = x_2$ and acted upon by the internal bending moment

$$M(x_2) = \int_a^b (y-x_2)\bar{p}(y)dy \quad (52)$$

The latter is due to $\bar{p}(y)$ distributed over the interval [a, b], to the right of $x = x_2$.

Study case (iii)

In this case, the beam response may be expressed as follows

$$u(x) = \int_a^b g_u(x,y)\bar{p}(y)dy + V \cdot g_u(x, x_V) \quad (53)$$

where V is the unknown reaction of the roller support at $x = x_V$. It can be computed by setting the compatibility condition at $x = x_V$

$$u(x_V) = \int_a^b g_u(x_V,y)\bar{p}(y)dy + V \cdot g_u(x_V,x_V) = -V \cdot \psi^V \quad (54)$$

which yields

$$V = -\left[\psi^V + g_u(x_V,x_V)\right]^{-1}\int_a^b g_u(x_V,y)\bar{p}(y)dy \quad (55)$$

Upon replacing Equation (55) for V in Equation (53), the deflection response $u(x)$ is given as

$$u(x) = \int_a^b g_u(x,y)\bar{p}(y)dy - g_u(x,x_V)\left[\psi^V + g_u(x_V,x_V)\right]^{-1}\int_a^b g_u(x_V,y)\bar{p}(y)dy \quad (56)$$

It is now worth remarking that Equations (38), (42) and (56) are closed-form solutions given in terms of the full set of beam parameters. The students may then implement these equations in a symbolic package and use it for different purposes. For instance, they could perform a sensitivity analysis where the influence of a given parameter (location and flexibility of the internal spring; location and flexibility of the external roller support; amplitude of the load interval $[a, b] \ldots$) is assessed on the deflection response at a given location. Other applications may be suggested, as already discussed.

CONCLUDING REMARKS

EB stepped beams with internal springs, subjected to static loads, have been given a closed-form solution based on pertinent Green's functions of the discontinuous beam. For these few relatively simple concepts the theory of generalized functions has been used. Solutions apply for arbitrary discontinuity patterns and arbitrary loads. Detailed expressions have been given for various sets of BC. The proposed solutions involve advantages with respect to competing methods in the literature, the most efficient of which involves enforcing at least four BCs [12].

It is believed that the proposed solutions may have a positive impact on teaching and learning. In fact no advanced knowledge of the theory of generalized functions is required and, at the expense of relatively hard mathematical derivations, engineering students are given a straightforward tool to tackle the bending problem of discontinuous beams, which play a key role in many engineering applications. It is also hoped that this paper may stimulate the interest of engineering students into the many potential applications of generalized calculus in mechanics.

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APPENDIX

Dirac delta and generalized functions

Among the generalized functions, the most used is the so-called Dirac delta function or *impulse* function, defined by

$$\int_{-\infty}^{\infty} \delta(x - x_0) dx = 1 \tag{A.1}$$

The above definition is generally explained by interpreting the Dirac delta in the following limit sense

$$\delta(x - x_0) = \lim_{\tau \to 0} \Pi\left(\frac{x - x_0}{\tau}\right) \tag{A.2}$$

where $\Pi((x-x_0)/\tau)$ is a unit-area rectangle function given as

$$\Pi\left(\frac{x-x_0}{\tau}\right) = \begin{matrix} 0 & x < x_0 - \frac{\tau}{2} \\ \tau^{-1} & x_0 - \frac{\tau}{2} < x < x_0 + \frac{\tau}{2} \\ 0 & x > x_0 + \frac{\tau}{2} \end{matrix} \tag{A.3}$$

In the light of Equation (A.3) any differential or integral operation applied on the Dirac delta function may be thought of as the limit, for $\tau \to 0$, of the same operation applied on the rectangle function $\Pi((x-x_0)/\tau)$ [8]. Equation (A.3) also explains the following properties of the Dirac delta, generally given as a complement to definition (A.1) [8]

$$\delta(x-x_0) = 0, x \neq x_0, \quad \int_{-\infty}^{\infty} \delta(x-x_0)\varphi(x)dx = \varphi(x_0) \tag{A.4a, b}$$

The Dirac delta plays a crucial role in modeling several phenomena in mechanics like, for instance, impulsive forces in dynamics.

In the theory of the generalized functions, the relationship between the Dirac delta function and the Heaviside function or *unit-step* function is fundamental; it is defined as

$$H(x-x_0) = \begin{matrix} 0 & x < x_0 \\ 1 & x > x_0 \end{matrix} \tag{A.5}$$

Specifically, bearing in mind Equation (A.3) it may be stated that

$$\int_{-\infty}^{x} \delta(s-x_0)ds = H(x-x_0) \tag{A.6}$$

and, in inverse form,

$$\frac{\bar{d}}{dx} H(x-x_0) = H_{,1}(x-x_0) = \delta(x-x_0) \tag{A.7}$$

Based on Equation (A.6) and Equation (A.7) the Heaviside function may be taken as the generalized integral of the Dirac delta and, conversely, the Dirac delta may be taken as generalized derivative of the Heaviside function (for this, the bar over the differentiation symbol is introduced). This will be always interpreted in the light of Equation (A.2). Recognize in fact that the limit

$$\lim_{\Delta x \to 0} \frac{H(x+\Delta x - x_0) - H(x-x_0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{H(x-(x_0-\Delta x)) - H(x-x_0)}{\Delta x} \tag{A.8}$$

is indeed the limit of a unit-area rectangle function of height Δx^{-1} spanning the interval $[x_0 - \Delta x, x_0]$. That is, exactly the definition (A.2) of Dirac delta.

Several other generalized functions may be introduced as generalized integrals of the Heaviside function. Among these the n-th order ramp function

$$R_n(x-x_0) = \begin{matrix} 0 & x < x_0 \\ \frac{(x-x_0)^n}{n!} & x > x_0 \end{matrix} \tag{A.9}$$

for n = 1,2. . . . , may be given as

$$R_n(x-x_0) = \int_{-\infty}^{x} R_{n-1}(s-x_0)ds = \underbrace{\int_{-\infty}^{x} \ldots \int_{-\infty}^{x} H(s-x_0)ds}_{n-\text{times}} \tag{A.10}$$

where $R_0(x-x_0) = H(x-x_0)$. Equation (A.10) may be also cast in the inverse form

$$\frac{\bar{d}}{dx} R_n(x-x_0) = R_{n,1}(x-x_0) = R_{n-1}(x-x_0), \quad \text{for } n = 1, 2 \tag{A.11}$$

For later convenience, the following generalized integrals are also of interest in this paper:

$$\int_{-\infty}^{x} f(s)\delta(s-x_0)ds = f(x_0)H(x-x_0) \tag{A.12}$$

which derives directly from Equation (A.6), and

$$\int_{-\infty}^{x} f(s)H(s-x_0)ds = H(x-x_0)\left[f^{[1]}(x) - f^{[1]}(x_0)\right] \tag{A.13}$$

where $f(x)$ is a function for which a first-order primitive $f^{[1]}(x)$ exists. To prove Equation (A.13) compute the generalized derivative of the r.h.s., that is [16]

$$\begin{aligned} &\frac{\bar{d}}{dx} H(x-x_0)\left[f^{[1]}(x) - f^{[1]}(x_0)\right] = \\ &= \delta(x-x_0)\left[f^{[1]}(x) - f^{[1]}(x_0)\right] + f(x)H(x-x_0) = f(x)H(x-x_0) \end{aligned} \tag{A.14}$$

Based on the same reasoning, it yields

$$\int_{-\infty}^{x} f(s)H(x_0-s)ds = H(x_0-x)\left[f^{[1]}(x) - f^{[1]}(x_0)\right] \tag{A.15}$$

and

$$\frac{\bar{d}}{dx} H(x_0-x)\left[f^{[1]}(x) - f^{[1]}(x_0)\right] = f(x)H(x_0-x) \tag{A.16}$$

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