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Teaching Quantitative Error Estimates for Engineering Approximations: Application to Torsion in Thin-Walled Sections*

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While engineering approximations are at the heart of engineering education and practice, students are rarely equipped with quantitative estimates of the errors associated with such approximations. Typically the curriculum includes only a qualitative discussion of the character of the errors involved and the assumptions made in using the studied approximate formulas or equations. Yet, as an engineer, the graduate student would often have to make decisions that depend on the level of accuracy of these approximations, e.g., a decision on the necessity to perform a costly computational analysis vs. relying on a standard approximate formula. The goal of this paper is to point to the need for in-class discussion on quantitative error estimates, as part of the engineering curriculum. As a case in point, the torsion of elastic rods with thin-walled cross sections is considered. Quantitative error estimates are provided for the standard formulas for the torsional stress and rigidity. A preliminary investigation is performed, involving 3rd-year students at the Department of Aerospace Engineering, Technion, at the end of a Structural Analysis course. This preliminary study shows that without having been exposed in class to quantitative error estimates, intuition leads most of the students to making the wrong practical decisions in some situations, which might have a negative impact on their future work as engineers. This study thus points to the educational benefit in teaching the subject of quantitative error estimates for engineering approximations during undergraduate studies. In the case of a 3rd year Structural Analysis course, the material associated with the subject would require about half an hour of frontal teaching, but can also be offered to the students as an enrichment in writing for self-study.

Keywords: engineering education; error estimates; solid mechanics; structural mechanics; torsion

1. Introduction

Engineering approximations are at the very heart of engineering education and practice. Examples are abundant and span all fields of engineering. Approximate and simplified theories, models and solutions to engineering problems are taught in practically any engineering course. Students are always made aware of the fact that such approximations are associated with errors, and they are expected to use them judiciously, and identify those cases in which using a certain approximate theory or solution might be invalid. Yet, students are rarely presented with *quantitative estimates* of the errors associated with such approximations.

Error estimates used in engineering practice can be divided into four categories:

1. *Error estimation as the central goal of the analysis*, as is typically done in closed-loop control theory. This category is not related to the subject of this paper.
2. *Error estimates for experimental measurements*. The basics of such error estimation (see, e.g., [1]) are taught to undergraduate engineering students in practically any university prior to their first laboratory course. There are important educational issues related to measurement errors; see, e.g., [2–4].
3. *Error estimates for numerical results obtained from computational schemes*. This is an extremely important subject that is closely related to the reliability of numerical results in engineering practice and to the notions of verification and validation [5–7]. However, due to its advanced nature, this subject is not usually taught in core undergraduate courses. In fact, error estimation of numerical results has been an active area of research for three decades and continues to be so; see, e.g., [8–10]. Advanced undergraduate courses devoted to computational methods (like a course on the finite element method) usually touch upon this subject.
4. *Error estimates for approximate analytical formulas used in engineering practice*. In this case the approximation is applied to a difficult mathematical problem and leads to a simplified problem that is much more amenable for analytic solution. When a closed-form solution can be found to the simplified problem, this results in a 'formula' that may have great practical

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importance for engineers in the relevant field of application. The error is a measure of the difference between the solution of the original problem and that of the simplified problem.

The latter type of error estimates is the focus of the present paper. It does not seem to receive the attention it deserves in undergraduate engineering textbooks and curricula. Students are rarely equipped with *quantitative* error estimates for standard engineering formulas studied as part of their undergraduate education. Typically the curriculum includes only a *qualitative* discussion of the character of the errors involved and assumptions made in using a given formula or equation. Yet, as an engineer, the graduate student will have to make decisions that depend on the level of accuracy of these approximations. As an example, the engineer will often have to make a decision as to whether it is necessary to perform a costly computational analysis or it is sufficient to rely on a standard approximate formula. Without having been exposed in class to quantitative error estimates during undergraduate studies, the engineer would have to rely solely on experience or intuition in this matter. Experience is often lacking in the first stages of an engineer's career, while intuition may be misleading, as will be demonstrated in this paper for a particular example. Other issues related to approximations and simplifications in engineering education have been dealt with previously in the literature; see, for example, [11–14].

The goal of this paper is to point to the need for in-class discussion on quantitative error estimates as part of the engineering curriculum, in cases where error estimates can be made available. To demonstrate this issue, an important subject in structural mechanics is considered, which is included in undergraduate courses of many engineering programs (and taught by the authors during the last few years), namely torsion of rods with thin-walled cross sections. This subject is part of any textbook on solid or structural mechanics—see, e.g., [15–18]–and has served in the past as a prototype and test-bed for various ideas in engineering education [19–22] . In particular, considered here are the standard approximate formulas for the torsional shear stresses and rigidity (or the angle of twist) for thin-walled members. These standard formulas are obtained from the three-dimensional St. Venant theory under the assumption that the thickness of the cross section is small with respect to the other length scales of the problem. Here quantitative error estimates are derived for these formulas.

This is a non-standard material, which is not to be found in most textbooks, yet the principle for deriving error estimates is quite simple and makes use of standard mathematical tools. The key to deriving error estimates is the identification of the most dominant term being neglected in the relevant equation (prior to simplification) and its comparison to the terms being retained. Students should be able to apply this principle to various simplified formulas after being exposed to one or two examples demonstrating it.

In order to substantiate the claim that quantitative error estimates should be taught as part of the undergraduate curriculum, a preliminary investigation was performed, involving 3rd-year students at the Department of Aerospace Engineering, Technion, who participated in a survey simulating a practical decision-making scenario. This investigation was conducted at the end of a Structural Analysis course taught by the two authors. This preliminary study shows that without having been exposed in class to quantitative error estimates, intuition leads most of the students to wrong practical decisions in some situations, which would potentially have a negative impact on their work as engineers. This study thus points to the educational long-time benefit in teaching the subject.

In the next section relative error estimates for torsion of uniform rods are derived in four different cases: (1) thin rectangular cross section; (2) thin-walled 'open' cross sections; (3) thin-walled 'closed' (mono-cell) cross sections; and (4) thin-walled multi-cell cross sections. The error estimates for these cases are obtained in different ways, each one being convenient for the case in point. Errors in cases (1) and (2) are estimated by considering the truncation error of an exact infinite series obtained for a rectangular cross section under the St. Venant theory. An error estimate for case (3) is obtained by deriving a two-term approximate solution of which one of the terms is neglected. Case (4) is handled by considering the coupled system of equations based on the formulas of case (3). In each case an error estimate is derived for the torsional shear stress and for the angle of twist (or for the torsional rigidity, which is inversely proportional to the angle of twist). Section 3 includes the description of the preliminary investigation mentioned above, via a student survey, and a discussion of the results. In the last section, some conclusions are drawn and future work is outlined.

2. Error estimates for torsional formulas

Consider a long uniform beam made of a linear elastic homogeneous and isotropic material, with a thin-walled cross section. The beam deforms and develop stresses due to *torsion*. It will be assumed that the beam is slender enough to behave according

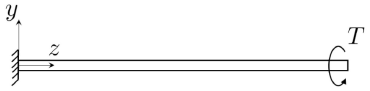

Fig. 1. A long uniform cantilever beam, with a thin-walled cross section. The beam deforms and develops stresses due to torsion.

to the St. Venant theory of beam torsion. Approximations that rely on the thinness of the cross section will be considered, and the central question will be: *how thin should the section be for these approximations to be sufficiently accurate?*

Let L be the length of the beam, and G be the material's shear modulus. A Cartesian system of axes (x, y, z) is introduced, where z is the longitudinal axis and (x, y) are the cross sectional axes. For the sake of simplicity of the presentation, the beam will be considered to be clamped at the left end $(z = 0)$ and be acted upon by a torsional moment T at the right end $(z = L)$. See Fig. 1. The results below can easily be extended to more general cases.

In deriving the error estimates in the following sections, use will be made of the elementary theory of calculating composite errors from basic errors; see, e.g., [1]. For example, if v_1 and v_2 are two variables, associated with absolute errors $E[v_1] \geq 0$ and $E[v_2] \geq 0$ and relative errors $e[v_1] \equiv E[v_1]/|v_1|$ and $e[v_2] \equiv E[v_2]/|v_2|$, then the errors associated with the *sum* and *product* of these variables are given by the formulas

$$E[v_1 + v_2] = E[v_1] + E[v_2] \ , \tag{1}$$

$$e[v_1 + v_2] = |v_1/(v_1 + v_2)|e[v_1] + |v_2/(v_1 + v_2)|e[v_2] \ , \tag{2}$$

$$e[v_1 v_2] = e[v_1] + e[v_2] \ . \tag{3}$$

In addition, for a variable v,

$$e[1/v] = e[v] \ . \tag{4}$$

All the equalities above should be understood in the asymptotic sense, under the assumption of small errors. This theory is usually taught to students in preparation for laboratory work, but here use will be made of it in the context of analytic approximations. In the sequel, e will always denote a positive relative error, i.e., $e = |v_{exact} - v_{approx}|/|v_{exact}|$, where v_{exact} and v_{approx} are the exact and approximate values, respectively.

2.1 Torsion of a rod with a thin rectangular cross section

First, consider a thin rectangular cross section, with side lengths t (thickness) and b (width), under a pure torsional moment T (see Fig. 1). The thinness of the cross section is expressed by $b \gg t$. The case of a thin rectangular cross section is important in that it

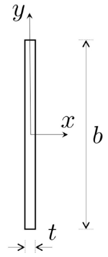

Fig. 2. A thin rectangular cross section with aspect ratio b/t. The standard formulas assume that $b/t \gg 1$ and neglect all terms except the dominating ones in the original expressions derived from the St. Venant theory.

constitutes the basis for calculating stresses and angles of twist for general thin-walled open cross sections (Section 2.2). Assume x to be the thickness direction, and thus $x \in [-t/2, t/2]$. See Fig. 2. The classical approximate formulas for the torsional shear stress τ, its maximum $\tau_{\max}$ and angle of twist α (per unit length) for this cross section are (see [15–18]):

$$\tau(x, y) = \frac{2Tx}{J} \quad , \quad \tau_{\max} = \frac{Tt}{J} \ , \tag{5}$$

$$\alpha = \frac{T}{GJ} \ , \tag{6}$$

$$J = \frac{bt^3}{3} . \tag{7}$$

Here J is the torsional rigidity constant of the cross section, which is the geometrical factor in the torsional rigidity of the cross section, GJ. These approximate formulas may be obtained under the St. Venant theory of torsion, in two ways. One approach is to degenerate the governing (Poisson's) equation into a one-dimensional equation, under the assumption that the stress varies in the x direction much more rapidly than in the y direction [17]. In the second approach, the torsional problem for a general rectangular cross section is first solved exactly, and then this solution is reduced to the formulas above by considering the asymptotic limit $t \to 0$ [15, 16, 18].

To estimate the errors involved in the approximate formulas (5)–(7), consider the second approach mentioned above. The exact solution for $\tau_{\max}$, α and J is given by (see, e.g., [15]):

$$\tau_{\max} = \frac{\beta Tt}{J} \ , \quad \beta = 1 - \frac{8}{\pi^2} \sum_{n\,\mathrm{odd}} \frac{1}{n^2 \cosh(n\pi b/(2t))} \ , \tag{8}$$

$$\alpha = \frac{T}{GJ} \ , \quad J = \gamma bt^3 \ , \tag{9}$$

$$\gamma = \frac{1}{3} \left(1 - \frac{192t}{\pi^5 b} \sum_{n\,\mathrm{odd}} \frac{\tanh(n\pi b/(2t))}{n^5} \right) . \tag{10}$$

Clearly, the approximate formulas (5)–(7) are obtained from the exact formulas (8)–(10) by taking only the leading term in the expressions on the right sides of (8) and (10), yielding $\beta = 1$ and $\gamma = 1/3$.

All textbooks that obtain both exact and approximate formulas justify the latter by pointing to the fact that the series in (8) and (10) converge very fast even for moderate aspect ratios b/t. For example, even for the extreme case of a square cross section ($b/t = 1$), three terms in the expression on the right sides of (8) and (10) (including the leading constant) suffice to yield the correct values of β and γ to three significant digits ($\beta = 0.675$ and $\gamma = 0.141$). Moreover, the larger the aspect ratio b/t is, the better the accuracy is, when the series is truncated after a fixed number of terms. The tables and graphs presented in [15, 16, 18] for the values of β and γ as functions of b/t suggest that a cross section may be regarded as 'thin' even if b/t is only moderately large, say $b/t \geq 5$.

While this information is closely related to the accuracy of the approximate formulas (5)–(7) , it does not provide the engineering student with explicit estimates of the errors generated by using these formulas. The latter information becomes important when considering general thin-walled open cross sections (Section 2.2), since the formulas used in that case are based directly on (5)–(7) . Table 1 shows the relative errors in the approximate values of $\tau_{\max}$ and J, obtained from (5)–(7), for different cross sectional aspect ratios. The '$\rightarrow 0$' appearing in the table denotes an error smaller than the machine precision for the computer used for this investigation (which is a Unix-based SGI/O2 workstation).

As Table 1 shows, there is a great difference between the errors generated by the approximate formula for the shear stress $\tau_{\max}$ and that for J, or for the angle of twist α. (The relative errors in J and in α are the same, from (6) and (4) .) When $b/t = 5$ the approximate formulas produce a 0.1% stress error but a 14% angle error. For $b/t = 10$ the stress error is completely negligible, but one still has a 7% angle error. Only for $b/t = 60$ one obtains an error of about 1% for J or α. These observations do not contradict the fact, mentioned earlier, that the series in (10) converges very fast. A little as three terms in the exact expansion (10) are always sufficient for excellent accuracy, but a single term may generate a large error!

Table 1 provides an answer to the question 'how thin is thin enough'. For example, if one is interested in calculating the stresses only, with an error tolerance of 3%, an aspect ratio of 3 is certainly enough to be considered 'thin'. However, if one is interested in calculating the angle of twist (e.g., in the case where the beam represents an aircraft wing, and the goal is to calculate the change in the angle of attack due to torsion), then, with the same level of accuracy, only an aspect ratio larger than 20 is 'thin' enough.

Table 1. Torsion of a thin rectangular cross section. relative errors in the approximate values of $\tau_{\max}$ and J (or α), obtained from (5)–(7), for different cross sectional aspect ratios

b/t	Error in $\tau_{\max}$	Error in J or α
1.0	48%	137%
2.0	8%	46%
3.0	1.5%	27%
4.0	0.3%	19%
5.0	0.1%	14%
10.0	$2{\cdot}10^{-5}$%	7%
20.0	$\rightarrow$ 0%	3%
60.0	$\rightarrow$ 0%	1.1%

Rough error estimate formulas can be obtained from the second term in (8) and in (10), which is the dominating term that is *not* taken into account in the approximate formulas (5)–(7). The relative errors for $\tau_{\max}$ and J are estimated in this way, respectively, by

$$e_\tau \simeq 1.6e^{-1.6(b/t)} , \tag{11}$$

$$e_J \simeq \frac{0.6}{b/t} . \tag{12}$$

In obtaining these expressions exponents of large negative arguments were neglected with respect to exponents of large positive arguments. These error estimates provide reasonable approximation to the exact error values given in Table 1.

2.2 Torsion of rods with thin-walled open cross sections

Consider a thin-walled open cross section of a general shape. See Fig. 3. The cross section is composed of M members, which are assumed for simplicity to be straight. Each member $m = 1, \ldots, M$ has width (length) b_m and constant thickness t_m.

Analysis of such cross sections under torsion [15–18] is based on the observation (which can be shown using the membrane analogy; see, e.g., [15]) that, except in the close vicinity of the joints connecting the section's members, the local stress field in each member is approximately that of a thin rectangular cross section. Once this is established, it is easy to show that the torsional rigidity is the sum of the

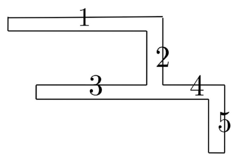

Fig. 3. A thin-walled open cross section. The section is composed of a number of straight members, which are numbered consecutively. The standard formulas assume that the aspect ratio b_m/t_m for each member m is large enough to allow the keeping of only the dominating terms in all expressions.

torsional rigidities of all the members. Thus, one obtains the formulas

$$(\tau_{\max})_m = \frac{Tt_m}{J} \quad , \qquad m = 1, \ldots, M \ , \tag{13}$$

$$\alpha = \frac{T}{GJ} \ , \tag{14}$$

$$J = \sum_{m=1}^{M} J_m \equiv \sum_{m=1}^{M} \frac{b_m t_m^3}{3} \ . \tag{15}$$

Stress concentrations develop in the vicinity of the joints connecting the straight members (see, e.g., [15]). Stress concentration factors are usually calculated separately (or are taken from an existing table of such factors) and will not be of concern here.

The relations between these formulas and (8)–(10) imply the following (to first order, which is sufficient in the error estimation itself):

- The torsional stress relative error in an open cross section is about the same as the relative stress error discussed in Section 2.1 for each member separately.
- The absolute error in the torsional rigidity constant (or in the angle of twist) is about the same as the sum of the absolute errors for each member separately.

Regarding the relative error e_J in J or α, it is easy to see that it is the weighted sum (see (2)) of the relative J-errors in each of the members, namely

$$e_J = \sum_{m=1}^{M} \frac{J_m}{J} (e_J)_m \ , \tag{16}$$

where $(e_J)_m$ is the error in J_m for an independent rectangular member.

Using (11) and (12) as well as (15) and (16), one has

$$(e_\tau)_m \simeq 1.6 e^{-1.6(b_m/t_m)} \ , \tag{17}$$

$$e_J \simeq 0.2 \sum_{m=1}^{M} \frac{t_m^4}{J} \ . \tag{18}$$

The latter estimate reduces to a particularly simple form if the cross section has *uniform thickness*, $t_m = t$ for $m = 1, \ldots M$. In this case (18) becomes simply

$$e_J \simeq 0.6Mt/S = \frac{0.6}{b_{av}/t} \ , \tag{19}$$

where S is the total length of the cross section members, i.e., $S = \sum_{m=1}^{M} b_m$, and b_{av} is the average member length. The right side of (19) is analogous to the right side of the single-rectangle estimate (12).

Since the error estimates in this case are based on those of a thin rectangular cross section (Section 2.1), the same comment made previously applies here. In particular, angle of twist errors are typically much larger than stress errors, and one should take special care when applying the angle of twist formula to open cross sections which are not extremely thin.

2.3 Torsion of rods with thin-walled mono-cell cross sections

Consider a thin-walled closed mono-cell cross section of a general shape. See Fig. 4. The cross section is composed of M members, which are again assumed for simplicity to be straight. Each member $m = 1, \ldots, M$ has width (length) b_m and constant thickness t_m.

The approximate formulas for the torsional stress and angle of twist in this case (see [15–18]) are:

$$\tau_m = \frac{T}{2\hat{A} t_m} \quad , \qquad m = 1, \ldots M \ , \tag{20}$$

$$\alpha = \frac{T}{GJ} \ , \tag{21}$$

$$J = \frac{4\hat{A}^2}{\sum_{m=1}^{M} b_m/t_m} \ . \tag{22}$$

Here $\hat{A}$ is the area enclosed by the cross sectional contour.

To obtain error estimates for these formulas, consider the way in which they are derived. There are a number of ways to obtain the expression (20) for the torsional stress. One of them, which is not the most common (see [15]), is based on showing that the stress τ can be written, under the St. Venant theory, as a superposition of two solutions: a solution τ_0 to an open-section problem and a solution τ_1 to a complementary problem. More precisely, the shear stress can be written as

$$\tau(s, u) \equiv \tau_0 + \tau_1 \simeq \frac{T}{J} \left(2u + \frac{\kappa}{t(s)} \right) \ . \tag{23}$$

Here κ is a constant, s is the arc-length coordinate along the perimeter of the section, and $u \in [-t(s)/2, t(s)/2]$ is the local coordinate orthogonal to s, in the thickness direction (see Fig. 4).

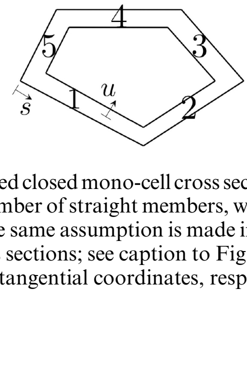

Fig. 4. A thin-walled closed mono-cell cross section. The section is composed of a number of straight members, which are numbered consecutively. The same assumption is made in this case as in the case of open cross sections; see caption to Fig. 3. The u and s are local normal and tangential coordinates, respectively.

Note that the part corresponding to τ_0 on the right side of (23) has the same form as (5) for a thin rectangular section. Both terms on the right side of (23) are the leading terms of exact infinite expansions.

Now, it can be shown that $\kappa = O(Rt)$ where R is a typical 'diameter' of the cross section, and t here is a typical thickness (say the maximum of $t(s)$). Hence the first term in the parentheses on the right side of (23), which is $O(t)$, is much smaller than the second term, which is $O(R)$. Therefore the first term is considered negligible with respect to the second. Additional analysis (without any additional approximation) shows that $\kappa = J/(2\hat{A})$, and this brings the second term to the final form which is given by (20).

In view of this derivation, the main approximation being done here in obtaining (20) is in neglecting τ_0 with respect to $\tau_1 = \tau_m$ in (23). This provides one with a stress error estimate, namely

$$(e_\tau)_m = \frac{(\tau_0)_{\max}}{\tau_m} \simeq \frac{Tt_m/J}{T/(2\hat{A}t_m)} , \qquad (24)$$

which yields, upon substituting (22) ,

$$(e_\tau)_m \simeq \frac{t_m^2 \sum_{k=1}^{M} b_k/t_k}{2\hat{A}} . \qquad (25)$$

In the particular case where t is uniform in the cross section, this estimate reduces to

$$e_\tau \simeq \frac{tS}{2\hat{A}} , \qquad (26)$$

where S is the perimeter of the cross section.

To derive an error estimate for α given by (21), consider the way (21) and (22) are obtained. First, using one of various techniques [15–18] , the relation

$$\alpha = \frac{1}{GT} \int_C \tau^2 t\, ds \qquad (27)$$

is derived. Here C is the closed contour enclosing the cross section. Substituting (20) in the integral, the formulas (21) and (22) follow. Using the relation (27) one can connect between the α-error and the τ-error. Denoting the exact stress and angle by $\bar{\tau}$ and $\bar{\alpha}$, respectively (as opposed to the approximate ones which are denoted simply τ and α), one can write

$$\begin{aligned} \bar{\alpha} &= \frac{1}{GT} \int_C \bar{\tau}^2 t\, ds = \frac{1}{GT} \int_C (\tau + \tau e_\tau)^2 t\, ds \\ &\simeq \frac{1}{GT} \int_C \tau^2 (1 + 2e_\tau) t\, ds \\ &= \alpha + \frac{2}{GT} \int_C \tau^2 e_\tau t\, ds . \end{aligned} \qquad (28)$$

From this, one may deduce

$$e_\alpha = \frac{\bar{\alpha} - \alpha}{\alpha} = \frac{2}{GT\alpha} \int_C \tau^2 e_\tau t\, ds . \qquad (29)$$

By replacing the integral by a sum over the straight members of the cross section, substituting the expressions (20)–(22) in (29) and simplifying, one finally obtains

$$e_\alpha \simeq \frac{\sum_{m=1}^{M} t_m b_m}{\hat{A}} . \qquad (30)$$

For a cross section with uniform thickness, this reduces to

$$e_J = e_\alpha \simeq \frac{tS}{\hat{A}} , \qquad (31)$$

namely *twice as large as the stress error* (cf. (26)).

As an example, consider a circular cross section of radius R and uniform thickness t. For this case one obtains from (26) and (31), $e_\tau = t/R$ and $e_\alpha = 2t/R$. Another example is that of a square cross section with side length a and uniform thickness t. From (26) and (31) one has $e_\tau = 2t/a$ and $e_\alpha = 4t/a$.

2.4 Torsion of rods with thin-walled multi-cell cross sections

Consider a thin-walled multi-cell cross section of a general shape. See Fig. 5. The cross section is composed of N closed cells (in Fig. 5, $N = 4$), which are indicated by $j = 1, \ldots, N$. Each cell j consists of M^j straight members with widths (lengths) b_m^j and constant thicknesses t_m^j (for $m = 1, \ldots, M^j$).

The analysis in this case [15–17] is a generalization of that for a mono-cell section. Each cell j (for $j = 1, \ldots, N$) is associated with its own constant shear flow q^j. The torsional shear stress at a point belonging to a single cell j (like point A in Fig. 5) is $\tau^j = q^j/t$, where t is the local thickness. At a point shared by two cells j and k (like point B in Fig. 5) the shear stress is given by $\tau^{jk} = q^{jk}/t \equiv (q^j - q^k)/t$. (For the sign convention, see, e.g., [16].) The shear

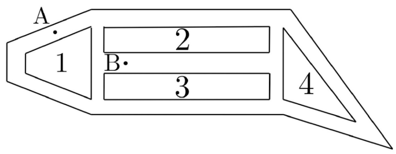

Fig. 5. A thin-walled multi-cell cross section. The section is composed of a number of closed cells (four in this figure). Each cell consists of a number of straight members. Some points in the section (like point A) belong only to one cell, whereas others (like point B) are shared by two cells.

flow at a point associated with cell j and member m is denoted q^j_m; it is either q^j if the point is shared by a single cell, or it is the appropriate $q^{jk} \equiv q^j - q^k$ if the point is shared by two cells.

The N shear flows q^j ($j = 1, \ldots, N$) are thus the primary unknowns of the problem. They are found by solving a coupled system of N equations:

$$2\sum_{j=1}^{N} \hat{A}^j q^j = T \ , \qquad (32)$$

$$\frac{\sum_{m=1}^{M^j} (b^j_m/t^j_m)q^j_m}{2G\hat{A}^j} - \frac{\sum_{m=1}^{M^{j+1}} (b^{j+1}_m/t^{j+1}_m)q^{j+1}_m}{2G\hat{A}^{j+1}} = 0 \quad , \quad j = 1, \ldots, N-1 \ . \qquad (33)$$

Here $\hat{A}^j$ is the area enclosed by cell j. Equation (32) expresses the fact that the torsional moments contributed from all the shear flows sum up to the resultant moment T. The $N-1$ remaining equations (33) come from the requirement that $\alpha^j = \alpha^{j+1} \equiv \alpha$ for all values of j, namely that all the cells rotate (around the same center of twist) with one common angle. Once the system (32)–(33) is solved and all the shear flows q^j are determined, the angle of twist $\alpha \equiv \alpha^j$ is found by evaluating the left-most term in (33) for any chosen cell j.

Deriving a sharp error estimate in this case is difficult. However, one can easily derive a conservative error estimate by making the following three observations. First, the assumptions that lead to the set of equations (32) and (33) are exactly those that underly the case of a mono-cell cross section. Therefore the errors estimated by (26) and (31) in the previous section are generated in the multi-cell case as well, for each of the cells separately. Second, all the equations (32)–(33) are fully *coupled*. This implies that the error associated with each of the variables is induced in all the other variables. Third, the set of $N-1$ equations (33) is associated with angles of twist even though the primary unknowns are the shear flows; this implies that the angle error (31) is induced in the shear flows, hence in the shear stress τ everywhere in the cross section. All these observations lead to the conservative error estimate (cf. (31))

$$e_\tau = e_\alpha \simeq \max_{j,m} \frac{t^j_m S^j}{\hat{A}^j} \ , \qquad (34)$$

where S^j is the perimeter of cell j, and the maximum is taken over all members of all cells.

3. Preliminary investigation and discussion

In order to substantiate the claim that quantitative error estimates should be taught as part of the undergraduate curriculum, a preliminary investigation was performed, involving third-year undergraduate students at the Department of Aerospace Engineering, Technion. This investigation was conducted at the very end of a Structural Analysis course that has been taught for several years by the two authors. As part of the course, the subject of pure torsion was taught over 4 weeks (2 weekly hours of lecture and 1 weekly hour of tutorial). The curriculum included, following the textbooks [15–18], a *qualitative* discussion on the errors involved, the assumptions made and the limitation associated with the torsion formulas. *No quantitative error estimates were taught.*

The students participated in a voluntary survey, simulating a practical decision-making scenario. The instructions were as follows (translated here from Hebrew):

> Imagine that you are an engineer in an industrial company, working on a project involving torsion of thin-walled beams. The project requires calculation of the shear stresses and torsional rigidity for two cross sections: a thin rectangular section of width b and thickness t, and a closed thin-walled square-shaped section of side length b and thickness t. You calculate the stresses and rigidity using the standard formulas that were studied in this course. Now you need to determine whether the results obtained by these formulas are reliable, or whether a detailed three-dimensional (3D) finite element analysis must be performed. This is an important decision, since on one hand you must obtain results that are accurate enough (see below), while on the other hand the 3D analysis is costly and would delay the progress of the project, and thus should be done only if it is really necessary.
>
> You have to consider the case in which a 10% accuracy level is desired for the calculated stress and rigidity values, and the case in which a 3% accuracy level is desired. Since there are two cross sections and two possible accuracy levels, altogether there are four cases (A, B, C, D), as indicated by the following four tables.[1] In each case, and for various values of the aspect ratio b/t, you are asked to determine whether a detailed 3D analysis is required. You have to write down 'yes' or 'no' in each cell of each of the tables. The answer 'yes' means that a 3D analysis is required, namely that the standard formulas would not provide the desired accuracy.
>
> Remark: In the case of the closed square cross-section, please ignore errors associated with stress concentrations at the corners. Assume that you are interested only in the stresses away from the corners.

Table 2 gives the estimated relative errors for stress and rigidity, for the two cross sections. The error values are given as a function of the cross sectional aspect ration, b/t, and are calculated using the estimates presented in Section 2. Based on these error values, the correct answer to the student survey can be deduced. It is indicated in Table 3 in

[1] See Table 3. The students received Table 3 with empty entries.

Table 2. Estimated relative errors for stress and rigidity, for thin rectangular cross section (up) and for closed thin-walled square cross section (down). The error values are given as a function of the cross sectional aspect ration, b/t, and are calculated from the estimates presented in Section 2; see Table 1, and equations (11), (12), (26) and (31) . These error values serve as the basis for the correct answer to the student survey given by Table 3.

	Closed thin square section	
	Stress error	**Rigidity error**
b/t	e_τ	e_J
2.5	80%	160%
4	50%	100%
10	20%	40%
100	2%	4%

	Thin rectangular cross section	
	Stress error	**Rigidity error**
b/t	e_τ	e_J
1	48%	137%
4	0.3%	19%
10	$2{\cdot}10^{-5}$%	7%
100	→0%	<1%

Table 3. Tables associated with the survey completed by 21 students at the end of the Technion's third-year aerospace engineering course 'Structural Analysis'. Students had to indicate 'yes' or 'no' in each entry of each table. An answer of 'yes' means that a detailed 3D analysis is required, namely that the torsion formulas studied during this course cannot be trusted to provide the desired level of accuracy. The correct answers are indicated in boldface. A * denotes those cases in which most of the students were wrong.

Case A: Thin rectangular cross section, 10% accuracy desired for τ	
b/t	Detailed 3D analysis required?
1	**Yes**: 21, No: 0
4	**Yes**: 5, No: 16 *
10	Yes: 1, **No**: 20
100	Yes: 0, **No**: 21

Case B: Closed thin square section, 10% accuracy desired for τ & J	
b/t	Detailed 3D analysis required?
2.5	**Yes**: 21, No: 0
4	**Yes**: 14, No: 7
10	**Yes**: 3, No: 18 *
100	Yes: 0, **No**: 21

Case C: Thin rectangular cross section, 3% accuracy desired for τ & J	
b/t	Detailed 3D analysis required?
1	**Yes**: 21, No: 0
4	**Yes**: 18, No: 3
10	**Yes**: 6, No: 15 *
100	Yes: 0, **No**: 21

Case D: Closed thin square section, 3% accuracy desired for τ & J	
b/t	Detailed 3D analysis required?
2.5	**Yes: 21**, No: 0
4	**Yes: 20**, No: 1
10	**Yes: 10**, No: 11 *
100	**Yes: 1**, No: 20 *

boldface. Table 3 also summarizes the data obtained from 21 students. All these students were in good academic standing and passed the course with a grade equivalent to A or B.

As seen from Table 3, the students correctly observed that the standard formulas become unreliable when b/t is too small. The difficulty was, of course, in determining how small b/t is allowed to be in each case. All students were successful in the extreme case of $b/t = 1$ or 2.5, where a detailed 3D analysis is always needed. The other extreme, $b/t = 100$, is more subtle since in cases A, B and C a 3D analysis is not needed, but in case D it is. This is implied by Table 2; the rigidity error is 4%, which is higher than the 3% accuracy level required. Almost all students were wrong in this case, but one may claim that the difference between 3% error and 4% error is too small to regard this as a serious failure.

More interesting are the student failures indicated in Table 3 by a * in cases A, B and C. In case A, with $b/t = 4$, the rigidity error is 19% as Table 2 shows, namely almost twice as large as the desired accuracy level of 10%. Yet, 16/21=76% of the students indicated that a detailed 3D analysis is not required. In case B, with $b/t = 10$, the rigidity error is 40%, i.e., four times the allowed error level, yet 18/21=86% of the students answered 'no'. In case C, also with $b/t = 10$, the rigidity error is 7%, more than twice the allowed error level of 3%, yet 15/21=71% of the students answered 'no'.

It is interesting to note that in all cases where most of the students were wrong (those indicated by * in Table 3), their mistake was in determining that a detailed 3D analysis is not required, when in fact it is. Thus, the students have generally erred in the 'dangerous' way, not in the over-conservative way.

This preliminary study suggests that without having been exposed in class to quantitative error estimates, intuition leads most of the students to wrong practical decisions in some situations, which might have a negative impact on their work as engineers. This study thus points to the educational benefit in teaching quantitative error estimates for engineering approximations during undergraduate engineering studies.

The investigation performed here is preliminary and limited in its ability to fully prove the claim made here, since it involved students that have not yet graduated, and did not include a control group. A more careful and complete investigation would include a survey involving young engineers of two different groups: those that were exposed to the subject of quantitative error estimates during their studies and those that were not (the control group).

4. Conclusions

In this paper it has been claimed that there is a need for in-class discussion of quantitative error estimates as part of the engineering curriculum. This material is non-standard and is rarely included in undergraduate engineering courses or found in textbooks that make use of approximate formulas or models, yet it may affect important decisions that the student will have to take as a young engineer, e.g., a decision on the necessity to perform a costly computational analysis vs. relying on a standard approximate formula. As a case in point, torsion of elastic rods with thin-walled cross sections was considered, and quantitative error estimates were provided for the standard formulas for the torsional stress and rigidity. In order to substantiate the paper's claim, a preliminary investigation was conducted, involving 3rd-year students. This study showed that, without having been exposed in class to quantitative error estimates, intuition led most of the students to wrong practical decisions in some situations. This suggests that teaching the subject of quantitative error estimates for engineering approximations during undergraduate studies has an important long-time benefit.

The amount of time required for teaching the subject of quantitative error estimates should be relatively very small. In the case of a 3rd year Structural Analysis course, the material associated with the subject would require about half an hour of frontal teaching. If time limitations prevent incorporating this additional material in the course plan, it can also be offered to the students as an enrichment in writing for self-study.

The investigation performed here was preliminary. A more careful and complete investigation should be conducted, that will include a survey involving young engineers of two different groups: those who were exposed to the subject of quantitative error estimates and those who were not (the control group). A better rate of success for the former group in coping with a decision-scenario simulation would strengthen this paper's claim.

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