International Journal of Engineering Education Vol. 38, No. 4, pp. 1010–1020, 2022
Printed in Great Britain
0949-149X/91 $3.00+0.00

Characterizing Back-of-the-envelope Problem-solving in Engineering*

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Back-of-the-envelope (BOTE) problems are informal estimation problems frequently used by engineers to come up with fast and simple estimates for solutions to much more complex problems. Although these problems are well-known within engineering, they remain essentially undefined in the academic literature. This poses a problem for instructors and managers who try to teach BOTE problem-solving, because one cannot teach something that one cannot measure or characterize. To this end, we interviewed 11 engineers from a range of subdisciplines about how and why they use BOTE calculations in their work. We chose to characterize BOTE calculations in terms of the cognitive decisions made by the solver at each stage of the process. We found that BOTE calculations could be characterized in terms of a limited set of 14 decisions. By identifying these decisions, we have identified the decisions one must practice and receive feedback on to learn how to solve BOTE problems.

Keywords: problem-solving; estimation; expertise

1. Introduction

There are frequently reports of skills gaps – disconnects between the knowledge graduating engineers have and the knowledge employers expect them to have once they enter the workplace [1]. This is because engineers receive relatively little practice solving the kinds of authentic engineering problems they will encounter in the workplace. This is typically limited to project-based, capstone and cornerstone design courses. Our previous work has shown that, though these courses do successfully teach students how to solve real engineering problems, students are far from where one might hope they would be at the end of an engineering program [2].

While the problems that engineering students encounter in most of their coursework are often difficult and involve complicated calculations or concepts, the solution is often procedural, and all information and necessary assumptions are given to the student. By contrast, authentic problems that students are expected to solve after college are unrestricted because "they possess conflicting goals, multiple solution methods, [non-scientific success standards, outside constraints], unanticipated problems, distributed knowledge, collaborative activity systems, and multiple forms of problem representation." [3]. We include examples of both kinds of problems in Table 1. The textbook problem presents all the necessary information, what assumptions can be made, and even what concepts apply. The authentic problems are open-ended, have multiple potential solutions depending on the assumptions made, require the solver to make a plan and gather information, and finally to make a decision based on the available data.

While, in practice, engineering problem-solving requires precision, engineers will frequently decide whether something is feasible via a more informal, back-of-the-envelope (BOTE) calculation. BOTE calculations are attributed to the physicist, Enrico Fermi, who was famous for posing challenging estimation problems that had a clever and brief solution (e.g., the piano tuner problem). The precise origin of this phrase is unknown, but Fermi is one of the most famous users of BOTE calculations. For example, he estimated the energy yield of the first atomic bomb knowing only how far some test pieces of paper were pushed by the blast waves [5].

Fermi's problems were usually designed to showcase approximation and simplification methods that can be used to solve interesting physics questions, like "How big an asteroid could you escape from by jumping?" [6]. These questions did not always have practical applications but modeled the type of simplifications that students should be learning and encouraged students to think carefully about how to model a problem before turning to a calculator. In practice, BOTE problems are more practical and have a specific purpose – to make a data-driven engineering decision without spending the time and resources to make a formal calculation.

Despite a careful literature review, we could find no academic definition of a BOTE calculation. The information that does exist is anecdotal and based on the experiences and ideas of individual physicists

* Accepted 23 January 2022.

Table 1. Example textbook problem from Ref. [4] vs. authentic problems with which practicing engineers would concern themselves

Textbook Problem
Tetrachloroethylene, $CCl_2{=}CCl_2$, is used widely as a dry-cleaning fluid and degreasing solvent. Tetrachloroethylene can be synthesized from carbon tetrachloride, CCl_4, by the following consecutive pyrolysis reactions at 800°C. $$2CCl_4 \rightarrow CCl_3{-}CCl_3 + Cl_2$$ $$CCl_3{-}CCl_3 \leftrightarrow CCl_2{=}CCl_2 + Cl_2$$ The first reaction is irreversible, but the reaction does not go to completion; there is CCl_4 in the reactor effluent. The second reaction is reversible and does not go to completion; there is hexachloroethane, $CCl_3{-}CCl_3$, in the reactor effluent. The Cl_2 from the first reaction impedes the forward progress of the second reaction. So rather than recycle $CCl_3{-}CCl_3$ to the reactor, the $CCl_3{-}CCl_3$ is separated and sent to a second pyrolysis reactor, but at 500°C. Consider the process flowsheet below. Use the chemical reactions above and the table of physical properties (pp. 76-78) to find at least three errors in the design. Ignore errors due to omitted pumps. Assume solid-gas separations are ideal; no gas leaves with the solid. Also, find two ways to improve the process, for example, by eliminating a separator and increasing the product yield. Ignore improvements by adding heat exchangers. Finally, change the process flow diagram to fix the errors and incorporate the improvements. Note: Solid $CCl_3{-}CCl_3$ is soluble in liquid $CCl_2{=}CCl_2$.
Authentic Problems:
1. Determine if it is feasible for this company to construct a manufacturing facility to produce Penicillin and make a profit. 2. The vapor flow rate of this distillation column is ½ of what is expected. Find out why and correct the issue.

and engineers [6–9]. In our personal conversations with educators and engineers, those who train engineers have expressed regret that their students are not learning how to do BOTE calculations in school. From an educational research perspective, this naturally prompts the question of "how do we teach this skill better?," but first we need to know what exactly the skills are that comprise BOTE calculations.

We chose to characterize BOTE calculations in terms of the decisions made by an engineer as they do a BOTE calculation. This is in line with arguments that problem-solving is about learning how to make decisions with limited information [10, 11]. We used retrospective interviews with practicing engineers to determine: (1) what is a BOTE calculation, (2) how is it used, and (3) how is its quality assessed.

2. Analytical Framework

Researchers in discipline-based education research have long studied problem-solving [3, 12–20]. Initial work was focused on identifying expert-novice differences in problem-solving [21], and later turned to develop methods of teaching problem-solving to students based on empirical findings and prescriptive models [22–27].

One important limitation to this previous work is that the results are based on how students and experts solve structured textbook-like problems. Though these are challenging for students, they are often procedural for experts. More recent work has looked at how experts solve problems that are challenging to them [15, 28]. These problems have conflicting goals, multiple solution methods, and multiple forms of representation [29]. Solving these problems involves making decisions with limited information, unlike problems typically encountered in courses [17, 30].

Our research group recently developed an empirical framework for how experts solve authentic problems. We conducted a study with over 50 expert scientists, engineers, and medical doctors to determine how they solved a typical problem in their work [31]. Interviews were based on a modification of the critical decision method of cognitive task analysis [32], and experts were asked to focus on the decisions that they made. We identified 29 decisions and 5 additional non-decision themes that were consistent across fields. Ref. [33] identified some of the decisions that Price et al., (2021) identified, but their list is not complete.

This previous work argued that this is a complete list of decisions involved in solving an authentic problem, and that the degree of authenticity of a problem could be characterized by the number of decisions left open to the solver and the number of decisions made for the solver. For example, a textbook problem allows the solver to make few decisions: what concepts apply, what similar problems has the solver seen before, and what calculations are needed. More authentic problems may require the solver to make assumptions, gather information, reflect on the quality of the information, etc. We thus sought to identify which decisions are involved in making BOTE calculations. In principle, BOTE calculations share many of the same characteristics as an authentic problem. The solver may have to decide how to model the system, what assumptions they can make, what information they need, whether information they have is valid/reliable, and whether their final solution makes sense. If we can identify which of the 29 decisions are involved in making BOTE calculations, we can then give students opportunities to deliberately practice making those decisions to strengthen their problem-solving skills [34].

3. Methods

We conducted semi-structured interviews with 11 expert engineers with at least 3 years of experience in industry to capture the decision-making process involved in solving authentic problems with BOTE calculations. The protocol was designed to first probe engineers' first impressions of BOTE calculations, and then explicitly ask them why they use those calculations, how they generally do them, and how they evaluate BOTE calculations. Then, following Ref. 31, we asked them to describe a recent BOTE calculation they had made. We made a methodological decision to use retrospective interviews following Price et al. [31]. This was chosen as an alternative to think-aloud interviews because we did not want to bias our findings by investigating how experts solved problems that we deemed to be BOTE problems, but which they may not have agreed fit that category. We discuss the limitations to our findings that result from this decision in the discussion. The interview protocol is given in subsection 3.1.

3.1 Interview Protocol

1. Are you familiar with "back-of-the-envelope calculations" or "Fermi problems"? If so, please briefly explain what you think defines these types of problems.
2. Why do you do back-of-the-envelope calculations?
   (a) What is the value of BOTE calculations?
   (b) What would make you decide to do a BOTE calculation?
   (c) Are there particular scenarios in which you would avoid doing a BOTE calculation?
3. What is the general approach that you take to back of the envelope problems?
   (a) What information do you need to know?
   (b) Do you think it is desirable to look up quantities or information for a BOTE calculation?
   (c) Do BOTE calculations require the use of physical law/models of physical phenomena? How can you decide what models to choose?
   (d) What is the role of physical/mathematical intuition in these problems?
4. How would you evaluate the quality of a back of the envelope solution?
   (a) How can you tell if the approximations and assumptions you are making are good?
   (b) How do you know if you are on the right path?
5. Please walk me through a recent back of the envelope calculation you made.
   (a) What decisions did you make?
   (b) What was the purpose of that calculation?
   (c) How did you decide if you were on the right path?

3.2 Analysis

The expert-made decisions were counted binarily: they either recalled having made a given decision in their professional experience or they did not. It is important to note that the engineers' work experience varied widely, including chemical engineering, biomolecular engineering, civil engineering, process-design engineering, materials science and engineering, and systems engineering. This means that, while there were central decisions made by all types of engineers, there may be slight differences in solving processes that are dependent on specific engineering subfields. These differences between branches of engineering were not meant to be captured with our interview protocol. We focused on what key decisions are shared among these different types of engineers to develop a stronger understanding of the solving process involved in making quick, order-of-magnitude estimations.

We used a grounded theory approach to analyze the interviews. We first examined the interview transcripts for evidence of themes in the data. We then came up with a preliminary coding scheme and hierarchical organization codes. After further discussion, we realized that our emergent codes could be re-labeled and re-organized according to the theoretical framework proposed by Price et al. [31]. We then re-coded the interviews using this *a priori* coding scheme. The 11 interviews were coded according to this *a priori* scheme by both authors. The two authors then met and resolved any disagreements in coding. Finally, the coding was reviewed by one of the authors of the original theoretical framework to check for misunderstandings.

Our group's expert-decision making framework was used to code the results from the interviews. This framework consists of 29 key decisions made by experts when solving authentic, real-world problems. These are decisions that students must practice, with the support and feedback from an instructor [34]. to develop expertise in problem-solving. They are categorized into six sections: (A) Selection and Goals, (B) Frame Problem, (C) Plan Process for Solving, (D) Interpret Information and Choose Solutions, (E) Reflect, and (F) Ongoing Knowledge and Skill Development. The expert-made decisions from our interviews were organized by these categorizations, however, the specific numbering and order of these decisions is solely for reference since, in practice, the decisions made by experts do not follow a repeated procedure.

4. Results

The number of engineers mentioning each decision is listed in Table 2.

Section A "Selection and Goals" consists of three decisions that experts make when determining the important problems within their field, the objectives to be accomplished within their problem, and the limitations of their task. 10/11 engineers said that they consider the goals, criteria, and constrains of the problem (Decision #3).

Section B "Frame Problem" includes six decisions that are made by experts when they are developing their solution plan and potential solutions. 10/11 subjects said that they considered the important features and information of their problem (#4). 11/11 said that they develop and use a predictive framework (#5). 7/11 said that they

Table 2. Decisions made by 9/11 or more engineers are labeled by two asterisks (**). Decisions made by 7/11 or more are labeled by one asterisk (*)

Category	Decision	X/11	Label
A. Selection and Goals	(1) What is important in field?	0	
	(2) Opportunity fits solver's expertise?	0	
	(3) Goals, criteria, constraints?	10	**
B. Frame Problem	(4) Important features and info?	10	**
	(5) What predictive framework?	11	**
	(6) Narrow down problem.	0	
	(7) Related problems?	7	*
	(8) Potential solutions?	10	**
	(9) Is problem solvable?	8	*
C. Plan Process for Solving	(10) Approximations and simplifications.	11	**
	(11) Decompose into sub-problems.	0	
	(12) Most difficult or uncertain areas?	0	
	(13) What info needed?	10	**
	(14) Priorities.	9	**
	(15) Specific plan for getting information.	0	
D. Interpret Information and Choose Solutions	(16) Calculations and data analysis.	11	**
	(17) Represent and organize info.	7	*
	(18) How believable is information?	7	*
	(19) Compare to predictions.	0	
	(20) Any significant anomalies?	0	
	(21) Appropriate conclusions?	0	
	(22) What is the best solution?	3	
E. Reflect (ongoing)	(23) Assumptions + simplifications appropriate?	3	
	(24) Additional knowledge needed?	0	
	(25) How well is solving approach working?	4	
	(26) How good is solution? How adequate is the chosen solution?	11	**
F. Implications and Communications of Results	(27) Broader implications?	0	
	(28) Audience for communication?	8	*
	(29) Best way to present work?	0	

consider problems that are related to their current problem (#7). 10/11 said that they consider the various solutions that are reasonable to expect when starting their problem (#8). 8/11 think about whether their problem is even solvable or worth solving (#9).

Section C "Plan Process for Solving" contains six decisions that are performed at the planning stage of the solution process. 11/11 experts use approximations and simplifications in their BOTE calculations (#10), 10/11 consider what information is required to adequately solve their problem (#13), and 9/11 said that they make it their priority to obtain a solution quickly and efficiently (#14).

Section D "Interpret Information and Choose Solutions" includes seven decisions that experts make when interpreting the reliability and validity of the available information and obtained solutions, respectively. 11/11 engineers performed calculations (#16), 7/11 represented and organized the information needed to solve the problem (#17), and 7/11 verified the believability of the information at hand (#18). 3/11 engineers had to figure out what the best solution out of a range of solutions was (#22).

Section E "Reflect" contains a set of 4 decisions that experts make throughout the entire solving process to reflect on their current work. 11/11 engineers determine if their solution is good or how good their solution is (#26). Less than 4/11 subjects reflected on the appropriateness of their assumptions and simplifications (#23) and on how well their solving approach is working for their current problem (#25).

Section F "Implications and Communications of Results" is built of three decisions that experts make when communicating information and results to their audience. 8/11 engineers considered who their audience for communication was when performing their BOTE calculation.

4.1 Examples from Experts

To illustrate the full range of decision making within subjects, we provide two annotated example BOTE calculations recounted by experts below.

Example 1:

I was working in a facility that was making electrolyte for batteries, and this was solid at room temperature. When it interacts with water it gives off acid, so we would make these in 50-gallon reactors, and at the end of the day you'd have a yield of a couple inches of sludge at the bottom that you had to get rid of. . . it had all the contaminants that you needed to throw out. The process was. . . they basically took a jackhammer. . . it's just really hard salt. . . you take a jackhammer and chip it out. You can imagine, if it's this water reactive stuff, it reacts with humidity in the air and gives off gas and stuff like. . . you give people a jackhammer and that's not a very good process **(#4 Important features and information)**. Someone said, what if we react it out with water, so you say ok. . . first order, what if I just take water, flow it in there, you know. . . how much water would I need to dissolve all this stuff **(#3 Goals and constraints)**.

The first order was like. . . I know the solubility, I know the numbers, so how much water would it take to dissolve this stuff. So, you say, alright, this is assuming 100% solubility, I don't have to do any sort of agitation or any sort of temperature stuff. Take the easiest case possible, if everything lines up and it goes according to plan, how much water do I need. . . It was some absurd amount, like 10,000 gallons per reactor **(#10 Approximations and simplifications, #16 Calculations and data analysis)**. I already knew, or I had to go find the environmental engineer, how much does it cost to dispose of that? So first, I say, it's like 10,000 gallons of water. . . it seems a lot of water for a 50-gallon reactor, but what does that cost? Because maybe it's pennies. He tells you, it depends, what pH is it? And, you know, your assumption. . . you have a very low pH, so that's hazardous waste **(#23 Assumptions and simplifications appropriate?)**. So, that's $6 a gallon. . . well ok, it can't be hazardous. So that's when I went into the baking soda and said, well what if I start neutralizing it? And then you start to build, kind of. . . and you follow this path and it's a lot of decision and judgement calls and things like that of. . . you know, how much do you know. . .. And the environmental engineer, he might say: well, you know we dispose of this other stuff similarly, and it costs this much **(#8 Potential solutions)**. So you say: how much do I really need an exact quote, right? The precision of your information, versus the quickness of getting it **(#14 Priorities)**, so. . . ok, we'll just take that similar thing because if it changes a little bit, it will be the same order of magnitude. And you start to just refine and refine and refine and figure out. . . what your model looks like. In that case, I determined that the way we had it, it was the cheapest **(#22 What is the best solution?)**. You know, you've got a known cost of the current process, and then I came up with the fact that we would need so much water and using the baking soda and all that, it just wasn't cost competitive with the current method. So yeah, there's a lot of just prioritizing the preciseness and availability of information, especially when, you know, you might need an answer today versus what's the cost of getting an answer next week (**#14 Priorities**).

Example 2:

We're trying to project our run rate for wafers through our equipment and basically the business folks were sitting in a meeting and they're like hey, here are the numbers. Here's how many wafers, you should be able to put out per hour through your equipment **(#8 Potential solutions?)**. I'm just looking at him, like that looks high, right like, I work on these tools. . . that looks high to me **(#18 How believable is information?)**. They're like, well, no, this is what was provided originally in the design phase. And I was like when? They were like 8 years ago. And I was like okay, let me think. And so, I just, like really simple stuff, just like ok I know how long each wafer takes to process, and there's a few different, you know, oh these, these take longer than these etc. but just like a rough weighted average of like. . . 30% of our material takes 5 minutes, 40% takes 10 minutes, something rough like that **(#4 Important features and info, #10 Approximations and simplifications)**.

[I told them] within an hour, I can put out like this many wafers and you said I can do 20% more, and I'm like you're over shooting **(#21 Appropriate conclusions?)**. And they're like oh interesting. So tomorrow I'm going to get follow up data to actually like prove it out with real runs, but it was at least enough to get star the conversation **(#25 How well is solving approach working?)**. Like hey, these numbers are off, so don't hold us to those if the business is projecting that we need this much material, we might need more equipment, right? Because they're under the wrong impression **(#21 Appropriate conclusions?)**. So, something like that took like 1 minute in the middle of the meeting just like hey you're probably off base we need to look at this a bit more.

5. Discussion

We categorized decisions that were made by 7/11 or more subjects as significant decisions and explain how they fit within the six categories of our problem-solving framework. In addition, we compare the results of expert engineers with those of a similar study performed on expert physicists.

5.1 Discussion of Which Decisions Were Found and Why

The only decision made by the expert engineers was deciding to consider the goals, criteria, and constraints of the problem (#3) to develop a predictive framework and a clear understanding of what specifically is being asked:

> ". . . the question that came from my processing is 'Should we fix this air fin fan? Are there credits to fix this fan?' "

These experts did not need to consider what is important in their field (#1) to develop a question or problem because they are either assigned problems or they select problems, based on their current task, to perform BOTE calculations on. Similarly, they do not need to consider if their expertise is sufficient to solve a BOTE problem (#2). Realistically, the solver decides before starting a BOTE calculation if the problem they were assigned is appropriate for their level of expertise and specialization.

Engineers in the interviews chose to frame their problem by identifying the important features and information of their problem (#4), what predictive framework, or mental model, is required to organize information and solve the problem (#5), any related work or problems that could help them (#7), the potential solutions that a problem might have (#8), which helps guide their solving process, and whether the problem is even solvable (#9) with a BOTE calculation or even a formal calculation. The following is an example of the previous five framing decisions made by a single engineer:

> "If you have a pretty good grasp of the concept at hand, if you know what variables are going to change and how or to what degree (#8), then it's a good time to run BOTE. If it's multivariable and you have less of an understanding of how changing one small variable will change the whole system then that's a pretty decent time to do the full calculation (#7). . . You should know the systems well enough to be comfortable with the uncertainty of the results you'll get (#7) with the model you picked. The model you pick depends on what you are working with, you use your familiarity with the system to help you pick an appropriate model (#4, 5)."

If an engineer has already moved to the stage of performing a BOTE calculation to answer a question or solve a problem introduced by their supervisor, co-worker, or customer, then they have already narrowed down the problem sufficiently (#6), thus explaining why this decision was not used in the interviews.

Our interview results show that expert engineers plan their solving process by considering what approximations and simplifications will be required to obtain a quick, approximate answer (#10), what information is necessary to perform a BOTE calculation (#13), and what aspects of the problem need to be prioritized (#14): "Time management is crucial (#14), so I need to figure out how precise my answer should be (#10), what info I have available to me (#13), and how long it would take to find that info."

The interviewed experts did not need to decompose their problem into sub-problems (#11) because their problem has already been simplified enough that they can perform a BOTE calculation, and they also did not need to identify the most difficult or uncertain areas of their problem (#12), since most frequently these calculations are performed on systems with which they are familiar (see quotes in comparison to physics section). In addition, the decision to make a specific plan for obtaining information (#15) was not performed in a BOTE calculation, since conducting an experiment or a survey to gather additional information would transform these informal calculations into formal calculations.

Following the collection of necessary information and development of a solving process, engineers perform these decisions when interpreting information and selecting a solution: making calculations (#16), determining how to use their previously established predictive framework to represent and organize their information (#17), and considering how believable or reliable their information is (#18):

> "The first order approximation (#16) was like: I know the solubility, I know the numbers, so how much water would it take to dissolve this stuff? So, you say, alright, this is assuming 100% solubility, I don't have to do any sort of agitation or any sort of temperature stuff. Take the easiest case possible, if everything lines up and it goes according to plan, then how much water do I need (#17)? It was some absurd amount, like 10,000 gallons per reactor. . . (#18)"

It was not necessary for our subjects to compare new information with previously made predictions (#19), because these experts plan their BOTE calculations by considering what information is needed to solve their problem, therefore they would not need to acquire any new information after starting their calculation. The only anomalies considered by our subjects (#20) were their potential solutions. As previously stated, if an engineer had to run an experiment to solve a BOTE calculation, then this approximate and informal calculation becomes rigorous and formal, thus explaining why no one in our interviews had to determine the appropriate conclusions of experimental data (#21) while performing a BOTE calculation.

When reflecting on the process and quality of their own BOTE calculation, engineers said that they only need to consider how acceptable their chosen solution is (#26):

> "He started changing the process to serve this purpose because he needed additional cooling. In this case, he was able to use that calculation to determine that, okay, we're probably good enough to reach that number, we're probably not going to be limited, and if we're limited it's going to be right around where we want it."

All other decisions in Section E "Reflect" were not significant in an expert engineer's solving process. These included making sure that their assumptions and simplifications were appropriate at various stages of their calculation (#23), if they needed additional knowledge that they did not already possess (#24), given their expertise and specialization. In addition, our subjects did not need to ensure that their solving approach was still working for their problem (#25). In the interviews, most engineers verified that their result made sense at the end rather than at various stages of their calculation, as these experts are often familiar with their systems and uncertainties that may arise.

Our interview results show that, when communicating their results and implications, these experts carefully consider who their audience for communication is (#28):

> "In order for you to have quality BOTE results, you need to have good communication between you and the recipient to see what their expectations are."

No engineer chose to consider what the broader implications of their results were (#27) and what the best way of presenting their work should be (#29). These engineers did not make the latter decision for their BOTE calculations since these calculations are inherently informal. An explanation for the absence of the former decision is detailed in our section "Comparison with Expert Physicists"

5.2 Summary and Limitations

Decisions made by 7/11 or more expert engineers are classified as significantly involved in the process of performing BOTE calculations. The following decisions are listed according to the order in which they appear in our group's expert decision-making framework: considering the goals, criteria, and constraints of the problem (#3), identifying the most important features and information (#4), considering what predictive framework to use (#5), using related problems or work (#7), considering the potential solutions to the problem (#8), figuring out if the problem is even solvable (#9), making approximations and simplifications (#10), figuring out what information is needed to solve the problem (#13), prioritizing efficiency (#14), performing calculations and data analysis (#16), representing and organizing available information (#17), considering the legitimacy of available data (#18), evaluating your chosen solution (#26), and considering your audience for communication of your results (#28). These decisions make up the "process" by which an expert engineer solves a BOTE problem, but in practice, problem-solving is characteristically iterative and non-linear [31].

These results provide evidence that prior experience solving authentic engineering problems may be necessary to be able to correctly formulate a BOTE

Table 3. Each decision cell contains (from left to right): Percentage of engineers ($x/11$) that made this decision, and percentage of physicists ($x/9$) that made this decision. Significant decisions made by both groups of experts are in **bold**. Differences between engineers and physicists over a magnitude of 50% are in **bold** and denoted by an asterisk **(*)**

A. Selection and Goals	(1) What is important in field?		(2) Opportunity fits solver's expertise?		**(3) Goals, criteria, constraints?**			
	0%	0%	0%	0%	91%	100%		
B. Frame Problem	**(4) Important features and info?**		**(5) What predictive framework?**		(6) Narrow down problem.		**(7) Related problems?**	
	91%	100%	100%	100%	0%	0%	64%	78%
	(8) Potential solutions?		**(9) Is problem solvable?**					
	91%	56%	73%	89%				
C. Plan Process for Solving	**(10) Approximations and simplifications.**		(11) Decompose into sub-problems.		(12) Most difficult or uncertain areas?		**(13) What info needed?**	
	100%	100%	0%	22%	0%	11%	91%	100%
	(14) Priorities.		(15) Specific plan for getting information.					
	82%	56%	0%	0%				
D. Interpret Information and Choose Solutions	**(16) Calculations and data analysis.**		**(17) Represent and organize information.**		**(18) How believable is information?**		(19) Compare to predictions.	
	100%	100%	64%	78%	64%	100%	0%	11%
	(20) Any significant anomalies?		(21) Appropriate conclusions?		(22) What is the best solution?			
	0%	0%	0%	0%	27%	11%		
E. Reflect (ongoing)	**(*23) Assumptions + simplifications appropriate?**		(24) Additional knowledge needed?		(25) How well is solving approach working?		**(26) How good is solution?**	
	27%	78%	0%	22%	36%	44%	100%	100%
F. Implications and Communications of Results	**(*27) Broader implications?**		**(*28) Audience for communication?**		(29) Best way to present work?			
	0%	67%	73%	11%	0%	0%		

calculation. Our interviews showed that many decisions regarding the framing of the problem are required for BOTE calculations. Students will not be able to do this unless they have opportunities to practice framing other engineering problems. This would seem to suggest that performing BOTE calculations requires some degree of expertise: experts perform BOTE calculations because they have the requisite skills to recognize how to formulate such problems.

There are a number of limitations to this work. First, this is an exploratory qualitative study with a small number of participants recruited from the authors' professional networks. It is thus not immediately clear whether these findings will generalize to a broader population of engineers, but this is an important first step in beginning to characterize estimation in engineering. Furthermore, we chose to have experts recount their understandings and experience with BOTE problems instead of using a think aloud protocol. This was done to remove some of our own personal bias as to what constitutes a BOTE problem in the findings. It may also be possible that experts may have been making decisions that were not reflected in their accounts of problem-solving. For example, we did not see experts identify what are important problems or questions in their field (#1), though it seems likely that this would happen during the process of an expert deciding to perform a BOTE calculation. Hammer and Elby [35] argue that epistemology is dynamic and context dependent. Thus, though the experts expressed a locally coherent set of decisions in their answers, this may be different from the set of decisions made in a more authentic context.

6. Comparison with Expert Physicists

In a recent study, we interviewed 9 expert physicists about how they use BOTE calculations [36]. we used the same decision-making framework to classify 15 decisions that appeared in 5/9 or more of their responses as significantly involved in their BOTE solving process. See Table 3 for a complete record of the differences between how many decisions were made by expert physicists and engineers. There were only 3 decisions in which both groups significantly differed, where there was a difference of 50% or more between the number of subjects that made a decision between both groups, and these were: (1) checking that their assumptions and simplifications were still appropriate (#23), where far more physicists than engineers made this decision, (2) considering what the broader implications

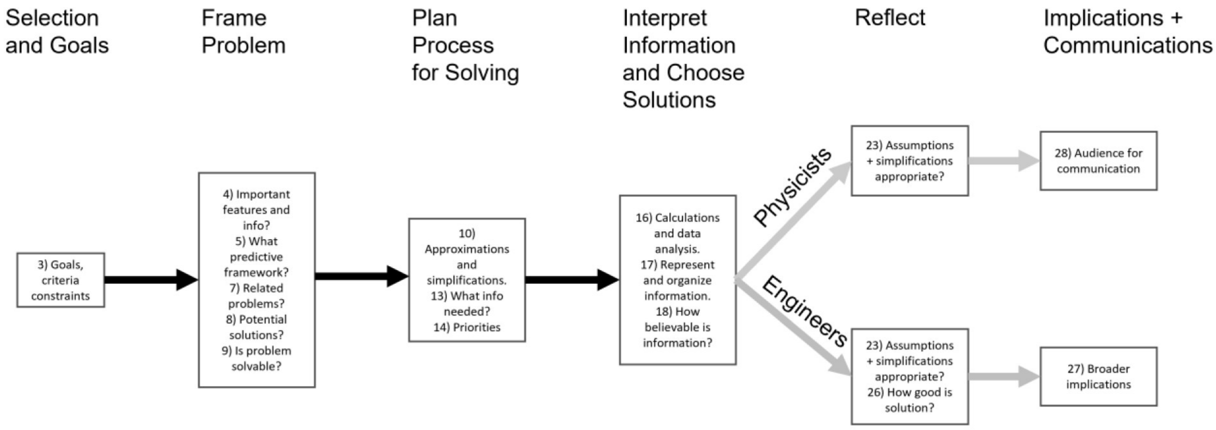

Fig. 1. Decision-making flow chart for engineers (bottom) and physicists (top). Decisions are presented in a logical order, but in reality the process is often iterative.

of their result are (#27), which far more physicists considered than engineers, and (3) choosing to consider who you will be communicating your results to (#28), where more engineers than physicists made this decision. We have summarized the decision-making process for physicists vs. engineers in Fig. 1.

These differences among engineers and physicists reveal how their goals and priorities can impact their use and application of BOTE calculations.

Table 4. Contains examples of three decisions that had significant differences in use between both groups of engineers and physicists. Quotes denoted with an asterisk (*) are used as examples for why that group of experts did not make that particular decision

Decision	Physicists	Engineers
(23) Assumptions + simplifications appropriate?	**Examples of physicists making this decision:** (1) "I look at the answer and try to assess whether or not it is something I might expect. I use my intuition to determine if my answer should have been smaller or larger, and I use my intuition to explain my answer. At all steps, you're always assessing if you're comfortable with what you're doing. Are all the steps reasonable? Or is there an assumption that you aren't as comfortable in?." (2) "Always try to reduce it to freshmen/sophomore physics- no matter what it is. Instead of reading the literature behind a physics phenomenon, I try to recreate the physics to better understand the problem."	**Examples of engineers not making this decision:** (*1) "I can tell that I am going on the right path if I have already solved a problem like this before. New systems are more prone to errors." (*2) "The initial values are all data, so I make sure that that the data are correct by checking their order of magnitude. I keep units throughout my work, so I can compare numbers together and make sure they make sense."
(27) Broader implications?	**Examples of physicists making this decision:** (1) "[Do BOTE calculations] before starting any problem to get a general feel for the physics of it." (2) "[By doing BOTE calculations,] students think about the physics before plugging in or looking up any equations, which develops their physics insight"	**Examples of engineers not making this decision:** (*1) "I'm more comfortable evaluating BOTE calculations from systems I've worked on before. I always back up my BOTE calculations with actual data and fact- I am more of a risk averse engineer. If you start getting BOTE problems wrong constantly then your reputation as an engineer starts to drop." (*2) "Generally I can tell [my calculations] are good because I'm solving the simplified version of a more complex problem that I have already solved multiple times. It's just through the intuition I've built over solving many problems."
(28) Audience for communication?	**Examples of physicists not making this decision:** (*1) "With physical intuition, a lot of times BOTE is used to build or check your physical intuition. Sometimes students plug in numbers quick and dirty when solving problems, so they don't get the intuition. These problems are useful for students to build their intuition." (*2) "[Use BOTE calculations] anytime you can. Sometimes thinking about all the formulas gets very complicated, so it's important to think about elementary ideas."	**Examples of engineers making this decision:** (1)"In order for you to have quality BOTE results, you need to have good communication between you and the recipient to see what their expectations are." (2) "[Use BOTE calculations] when you have to make a quick decision and you don't want to interrupt the flow of convo. This is dependent on who you are talking to, like a technician, engineer, or manager."

Engineers said that they often work with established models that they have practiced in their continuous study and work, while physicists are more likely to derive their own models using dimensional analysis and similar methods of approximation, so it makes sense that physicists would need to ensure that their assumptions and simplifications are valid and that engineers would already be comfortable using models and equations that they have practiced in the past. See Table 4 for examples of engineers being familiar enough with their systems that they do not need to constantly ensure that their model is working for their problem. Likewise, it would not be necessary to consider what the broader implications of a solution are for an expert engineer because their BOTE calculations are typically narrower in scope than those conducted by physicists. See Table 5 for contrasting examples of how physicists use BOTE calculations to develop a deep learning and understanding of the underlying physics concepts of their problem, while engineers do not need to think deeply about their result as their familiarity with the systems they are working with allows them to recognize and apply their practical result. An interesting difference among the two groups is that it is mainly the engineers that need to consider who their audience for communication is, as they need to work closely with their recipients, which includes teams and management, in order to deliver an answer that is quick to calculate and adequate for their given requirements.

7. Conclusions

This work represents a first step in defining the scope and purpose of BOTE calculations in engineering. Using the previously established expert decision framework, we have cast BOTE problems in terms of a finite set of skills to be practiced with feedback in order to gain competency in performing these calculations. These findings provide interesting opportunities for future research, such as measuring the impacts of REU programs on BOTE problem-solving ability in undergraduates. One could also conduct longitudinal studies with physics and engineering students to see how these skills develop through both coursework and research/internship experiences. We hope that this initial work will encourage other researchers studying problem-solving to more thoroughly investigate estimation and the important role it plays in engineering.

References

1. C. D. Grant and B. R. Dickson, Personal Skills in Chemical Engineering Graduates, *Education for Chemical Engineers*, **1**(1), pp. 23–29, 2006.
2. E. Burkholder, L. Hwang and C. Wieman, Evaluating the Problem-Solving Skills of Graduating Chemical Engineering Students, *Education for Chemical Engineers*, **34**, pp. 68–77, 2021.
3. D. H. Jonassen, Engineers as Problem Solvers, *Cambridge Handbook of Engineering Education Research*, Cambridge University Press, Cambridge, pp. 103–18, 2014.
4. T. M. Duncan and J. A. Reimer, *Chemical Engineering Design and Analysis: An Introduction*, Cambridge University Press, Cambridge, 2019.
5. M. Burgess, Eyewitnesses to Trinity, *Nuclear Weapons Journal*, **2**, p. 45, 2005.
6. E. Purcell, The Back of the Envelope, *American Journal of Physics*, **53**(7), pp. 615–615. 1985.
7. D. Dunn-Rankin, Evaluating Design Alternatives the Role of Simple Engineering Analysis and Estimation, *Proceedings, 2001 Annual ASEE Conference Proceedings*. ASEE, 2001.
8. A. Zee, *Fly by Night Physics: How Physicists Use the Backs of Envelopes*, Princeton University Press, Princeton, 2020.
9. S. Mahajan, Welcome to the Back of the Envelope, *American Journal of Physics*, **85**(12), pp. 959–61, 2017.
10. C. E. Wieman, Expertise in University Teaching & the Implications for Teaching Effectiveness, Evaluation & Training, *Daedalus*, **148**(4), pp. 47–78, 2019.
11. D. L. Schwartz and D. Arena, *Measuring What Matters Most: Choice-Based Assessments for the Digital Age*, MIT Press, Cambridge, 2013.
12. J. I. Heller and F. Reif, Prescribing effective human problem-solving processes: Problem description in physics, *Cognition and Instruction*, **1**, p. 177, 1984.
13. P. Heller and M. Hollabaugh, Teaching problem solving through cooperative grouping part 2: Designing problems and structuring groups, *American Journal of Physics*, **60**(7), pp. 637–44, 1992.
14. P. Heller, R. Keith and S. Anderson, Teaching problem solving through cooperative grouping part 1: Group versus individual problem solving, *American Journal of Physics*, **60**, p. 627, 1992.
15. W. K. Adams and C. E. Wieman, Analyzing the many skills involved in solving complex physics problems, *American Journal of Physics*, **83**, p. 459, 2015.
16. C. Atman, R. S. Adams, M. E. Cardella, J. Turns, S. Mosborg and J. Saleem, Engineering design processes: A comparison of students and expert practitioners, *Journal of Engineering Education*, **96**(4), pp. 359–79, 2007.
17. N. S. Hong, D. H. Jonassen and S. McGee, Predictors of well-structured and ill-structured problem solving in an astronomy simulation, *Journal of Research in Science Teaching*, **40**(1), pp. 6–33, 2003.
18. L. Hsu, E. Brewe, T. M. Foster and K. A. Harper, Resource letter rps-1: Research in problem solving, *American Journal of Physics*, **72**, p. 1147, 2004.
19. D. Jonassen, Design for decision making, *Educational Technology: Research and Development*, **60**, pp. 341–59, 2012.

20. C. E. Wieman, Comparative cognitive task analyses of experimental science and instructional laboratory courses, *The Physics Teacher*, **53**, p. 349, 2015.
21. M. T. H. Chi, P. J. Feltovich and R. Glaser, Categorization and representation of physics problems by experts and novices, *Cognitive Science* **5**, pp. 121–52, 1981.
22. G. Cabodi, P. Camurati, P. Pasini, D. Patti and D. Vendraminetto, D, Schema-based instruction with enumerative combinatorics and recursion to develop computer engineering students' problem-solving skills, *Int. J. Eng. Educ*, **36**(5), pp. 1505–1528, 2020.
23. J. L. Docktor, J. Dornfeld, E. Frodermann, K. Heller, L. Hsu, K. A. Jackson, A. Mason, Q. X. Ryan and J. Yang, Assessing student written problem solutions: A problem-solving rubric with application to introductory physics, *Physical Review Physics Education Research*, **4**, 010111, 2016.
24. E. W. Burkholder, J. K. Miles, T. J. Layden, K. D. Wang, A. V. Fritz and C. E. Wieman, Template for teaching and assessment of problem-solving in introductory physics, *Physical Review Physics Education Research*, **16**(1), 010123, 2020.
25. S. J. Kimmel, J. S. Kimmel and F. P. Deek, The common skills of problem solving: From program development to engineering design, *Int. J. Eng. Educ*, **19**(6), pp. 810–817, 2003.
26. N. J. Mourtos, Challenges students face in solving open-ended problems, *Int. J. Eng. Educ.*, **26**(4), pp. 846–859, 2010.
27. K. Heller and P. J. Heller, *Competent Problem Solver – Calculus Version*, McGraw-Hill, New York, 2000.
28. C. Singh, When physical intuition fails, *American Journal of Physics*, **70**, pp. 1103–9, 2002.
29. D. H. Jonassen, J. Strobel and C. B. Lee, Everyday problem solving in engineering: Lessons for engineering educators, *Journal of Engineering Education*, **95**(2), pp. 1–14, 2006.
30. B. Means, E. Salas, B. Crandall and T. O. Jacobs, Training decision makers for the real world, *Decision Making in Action: Models and Methods*, Ablex, Norwood, NJ, pp. 306–26, 1993.
31. A. M. Price, C. Kim, E. W. Burkholder, A. V. Fritz and C. E. Wieman, A detailed characterization of the expert problem-solving process in science and engineering: guidance for teaching and assessment, *CBE Life Sciences Education*, **20**(3), 2021.
32. B. Crandall, G. A. Klein and R. Hoffman, *Working Minds: A Practioner's Guide to Cognitive Task Analysis*, MIT Press, Cambridge, Massachusetts, 2006.
33. J. Harlim and I. Belski, Long term innovative problem solving skills: Redefining problem solving, *Int. J. Eng. Educ.*, **29**(2), pp. 280–290, 2013.
34. K. A. Ericcson, M. J. Prietula and E. T. Cokely, The Making of an Expert, *Harvard Business Review*, 2014.
35. A. Elby and D. Hammer, Epistemological resources and framing: A cognitive framework for helping teachers interpret and respond to their students' epistemologies, *Personal epistemology in the classroom: Theory, research, and implications for practice*, Cambridge University Press, Cambridge, pp. 409–434, 2010.
36. G. Murillo-Gonzalez and E. Burkholder, What decisions do experts make when doing back-of-the-envelope calculations? *Physical Review Physics Education Research,* **18**, 010125, 2022.

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