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Mode I Stress Intensity Factor by the Method of Caustics*

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This paper deals with the method of caustics which is an optical method used to study the effects of the presence of a crack in machines or structures. The method can be used to demonstrate to students (undegraduate and graduate) how much damage a crack can do to a machine part or to a structural member. Suggested experiments for undergraduate students to run in the laboratory are included.

INTRODUCTION

MACHINES, structures and devices do not last forever. However, some fail earlier than others for a variety of reasons. Fracture is one important reason for the failure of engineering designs. Fracture mechanics has grown to be a large field of study to help engineers deal with this type of failure. Normally, students are not exposed to fracture mechanics until they are in graduate school. Yet, in the United Stated, at least, many students go to work as engineers after completing their undegraduate studies rather than enroll into graduate school. For this reason, some aspects of fracture mechanics should be demonstrated to students in the laboratory whenever possible.

The presence of a crack in a material changes the distribution of stresses in the general neighborhood of the crack tip. The design engineer needs to know how much change the presence of a crack brings, where the changes occur, as well as their potential implications for the function for which the part was designed.

PRELIMINARY CONCEPTS

Consider a rectangular plate into which a crack is imbedded. The stress components in the vicinity of the crack due to an applied tensile load perpendicular to the crack axis are shown in Fig. 1, and are given by the modified Westergaard equations as:

$$\sigma_x = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left(1 - \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right) + HOST \tag{1a}$$

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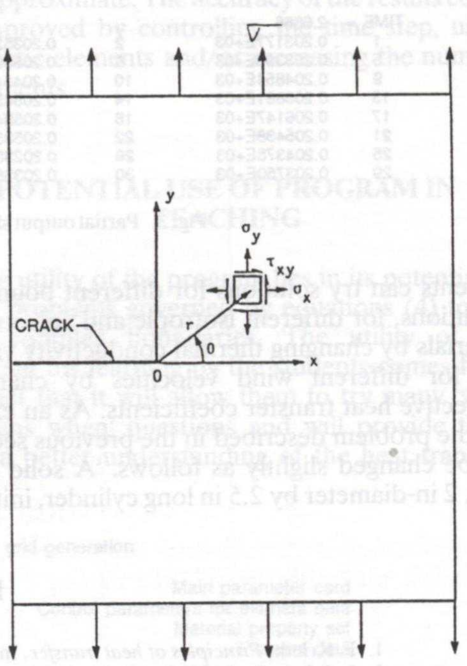

Fig. 1. Coordinate system with respect to the crack tip.

$$\sigma_y = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left(1 + \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right) + HOST \tag{1b}$$

$$\tau_{xy} = \frac{K_I}{\sqrt{2\pi r}} \sin\frac{\theta}{2}\left(\cos\frac{\theta}{2}\cos\frac{3\theta}{2}\right) + HOST \tag{1c}$$

Where K_I is mode I stress intensity factor (the opening mode) and is defined as $K_I = \sigma\sqrt{\pi a}$. σ is the applied stress and a is the crack length. HOST denotes higher order stress terms and r and θ are the polar coordinates with the origin at the crack tip as shown in Fig. 1. It was demonstrated that the shape of caustics is virtually unaffected by the presence of higher order terms and they can be neglected [1].

The essence of fast fracture is that it is a failure mechanism involving the unstable propagation of a crack in a structure. In other words, once the crack has started to propagate, the loading system is such that it produces accelerated growth. In the history of failure by fast fracture in service structures, fracture has always been produced by applied stresses less than the design stress calculated using the appropriate code and safety factor. This has naturally enhanced the catastrophic nature of the fractures and has led to the general description of them as being brittle. A brittle fracture is the start of an unstable crack propagation produced by an applied stress that is less than the yield stress of the uncracked ligament remaining when instability first occurs. Such brittle fractures are related to fracture parameters, called the stress intensity factors (SIF). The goal of engineers is to avoid fracture in structural and machine elements by calculating the SIF for a particular crack geometry and loading conditions and adjusting the design in an appropriate manner.

BRIEF HISTORY OF THE METHOD OF CAUSTICS

The method of caustics, also known as 'shadow spot method', in various investigations has proven to be a powerful method to measure SIF at a crack tip in static and dynamic fracture mechanics problems. In the method of caustics, all the information is obtained from the deformation in the close vicinity of the crack tip which is a region of much interest in fracture mechanics. The method is a relatively new experimental technique for determining SIF as compared to the traditional experimental stress analysis methods of electrical-resistance strain gages and photoelasticity. The method can be used to calculate the stress intensity factors at the crack tip quickly and accurately. Yet, the method of caustics is not covered in undergraduate experimental stress analysis textbooks (or graduate level). The simple way for the determination of SIF by the method of caustics permits the use of this method in undergraduate laboratory (or in practical applications) where cracked specimens are encountered.

The first attempt to use the caustics and their properties for studying singular fields in elasticity was made by Manogg in 1966 [2]. He developed the theory for transmitted mode I caustics. Theocaris in 1970 developed the technique where he used the reflected light from both the front and rear faces [3]. Rosakis *et al*. used the reflected caustics from non-transparent materials [4]. They determined mode I SIF (the opening mode) K_I by measuring the maximum transverse diameter of the caustic image obtained from optically isotropic materials. The method of caustics was extended to study mode I SIF by using optically anisotropic materials [5].

PHYSICAL PRINCIPLE OF THE METHOD OF CAUSTICS

The basic physical principle describing the method of caustics is shown in Fig. 2[6]. Due to high stress concentration in the region surrounding the crack tip, both the thickness and the refractive index of the transparent material change. As a consequence, the area surrounding the crack tip acts similar to a divergent lens and is called the initial curve. A monochromatic light beam emitted from a He-Ne laser impinges on the stressed crack specimen. Due to the presence of the lens effect very close to the crack tip, the reflected or transmitted light rays are deviated outwards. These deviated rays are concentrated along a strongly illuminated surface in space, which forms the caustic surface. When the caustic surface is projected on a screen in front of the specimen at a distance Z_0, a singular curve, called the reflected caustic is formed on the screen. If the caustic surface is projected on a screen behind the specimen at a distance Z_0, the transmitted caustic is formed on it. The schematic of the experimental setup for the reflected caustic is shown in Fig. 3. The transmission caustic setup is shown in Fig. 4. The caustic is the resulting image of the light beam transmitted or reflected from the divergent lens. The transmitted or reflected light rays are deviated outwards as shown in Fig. 2. Thus, both the crack tip and the initial curve cannot be seen on the caustic image. The location of the crack tip is obscured by the caustic. The reflected caustic from the front face is related only to the mechanical properties [E and ν] of the material. E is Young's modulus and ν is Poisson's ratio. The transmitted caustic and the reflected caustic from the rear face are related to both the mechanical and optical properties of the material [E, ν, and n] where n is the refractive index of the material.

The shape and dimensions of the caustic, which is always a generalized epicycloid curve, depend on the stress field singularity (the presence of the crack), material properties and experimental set up. Note that the method of caustics inherently provides information close to the crack tip where the theory of elasticity near-field solution (equation 1) is valid.

The experimental transmitted caustic through plexiglas is shown in Fig. 5 and the experimental reflected caustic is shown in Fig. 6. Neither the crack tip nor the initial curve can be seen on the experimental caustics as mentioned earlier. Furthermore, the experimental inner caustic, resulting from the reflection from the front face, is not closed. This is due to the crack opening displacement.

As mentioned before, K_I can be determined by measuring the maximum transverse diameter of the caustic image. Theoretically, the relevant caustic line should be defined by the transition from the dark inner region to the bright rim on the caustic pattern. Mode I stress intensity factor can be deter-

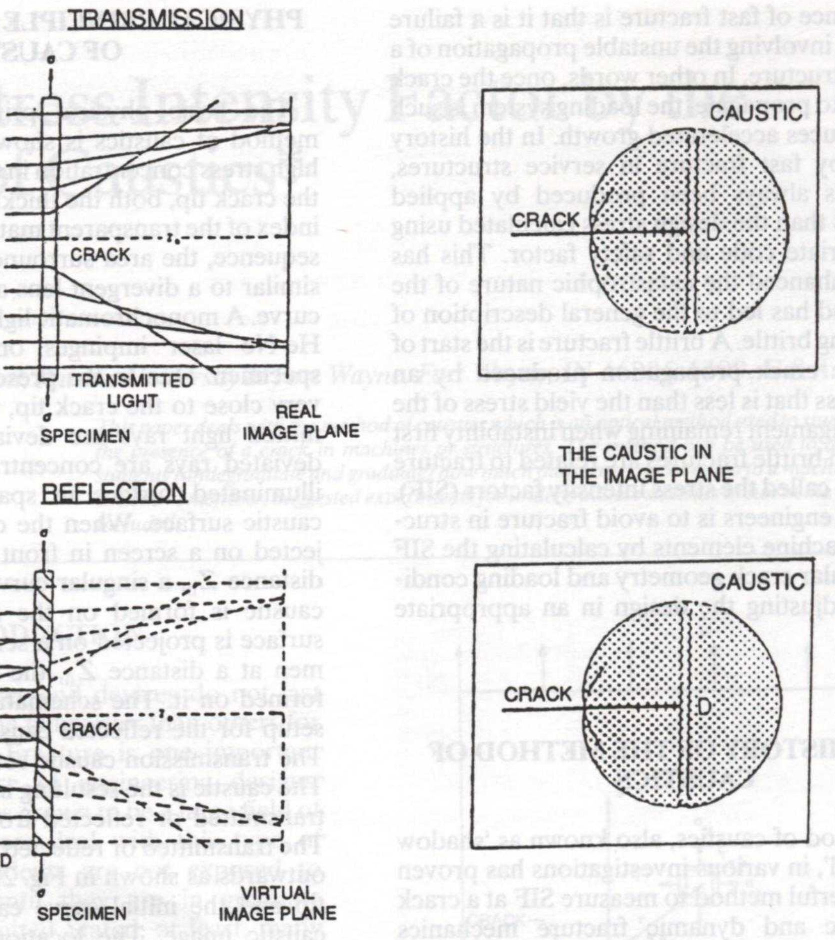

Fig. 2. The principle of the metod of caustics for transmission and reflection.

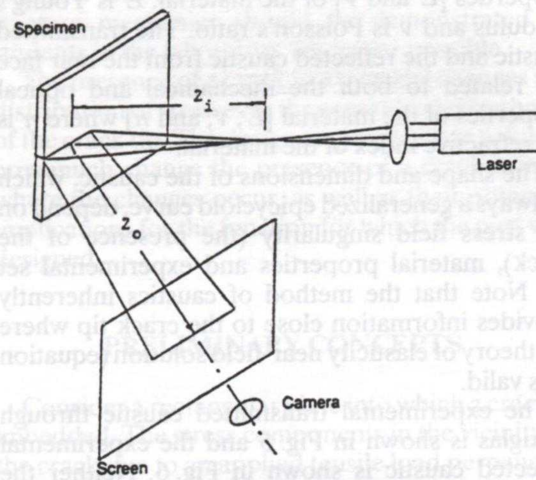

Fig. 3. Schematic reflected caustic setup.

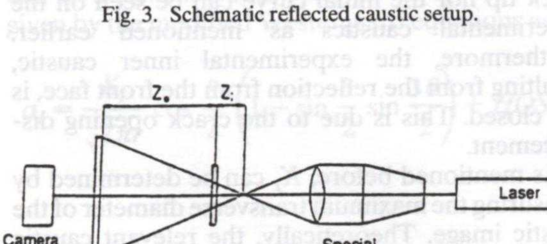

Fig. 4. Schematic transmitted caustic setup.

mined by using either the transmitted caustic or the reflected caustic. In other words, there is no need to use both caustics.

THE BASIC FORMULAS AND PROCEDURES FOR DETERMINING MODE I STRESS INTENSITY FACTOR

The reflected caustics from both the front and rear faces of mechanically and optically isotropic materials (plexiglas) are considered. The formulas for transmitted caustic through transparent material or reflected caustic from non-transparent material are the same except for the optical path changes. A light beam traverses the specimen at point $p(r, \theta)$ in the object plane as shown in Fig. 7. The non-deflected beam would pass the reference plane (the screen) at point P_m defining the vector $\mathbf{r_m}$. Due to the presence of the lens, the reflected light beam is displaced to point $p'(x', y')$ by a vector **w**. **w** is a function of the coordinates r and θ of point P. The vector $\mathbf{r}'$ of the image point P' is given by

$$\mathbf{r}' = \mathbf{r}_m + \mathbf{w}. \tag{2a}$$

When the light is slightly converging or diverging, the image size at the screen is not the same as that at

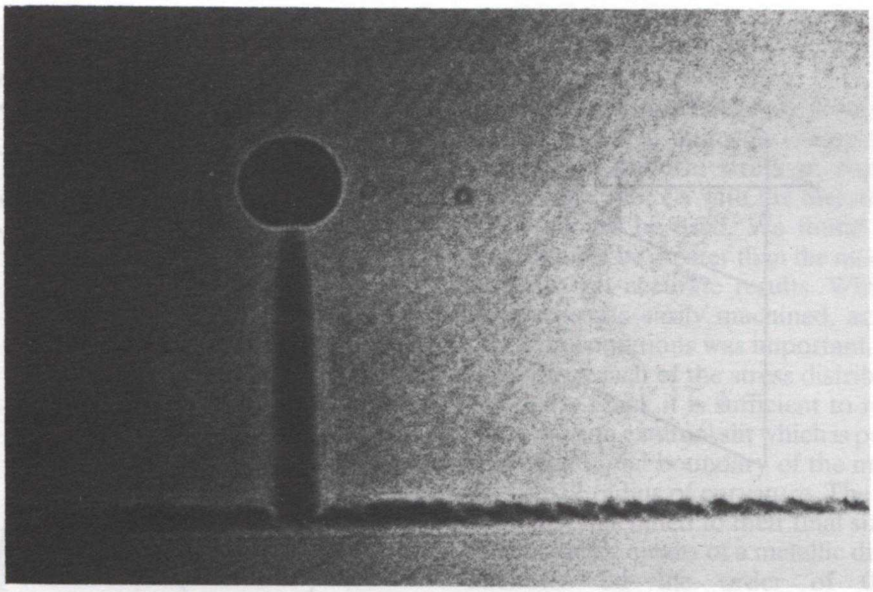

Fig. 5. Transmitted caustic through a transparent material.

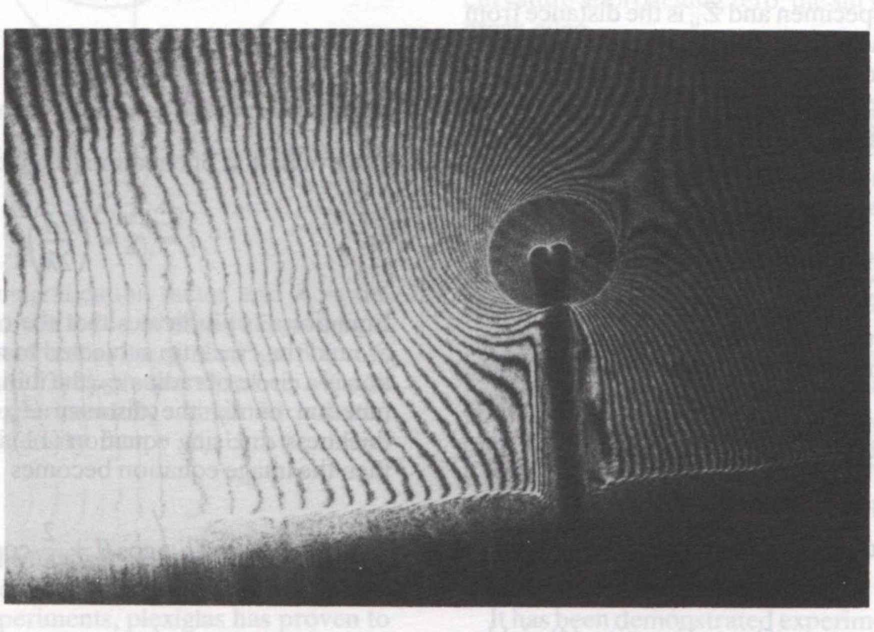

Fig. 6. Reflected caustic from a transparent material.

the model. If the image magnification factor is λ, then the vector $\mathbf{r}'$ of the image point p' becomes

$$\mathbf{r}' = \lambda \mathbf{r}_m + \mathbf{w}. \qquad (2b)$$

The caustic image (shadow optical image) is completely described by equation (2). For each point $p(r, \theta)$ in the vicinity of the crack tip, the corresponding image point p' of the caustic image is obtained. As an envelope, the caustic is a singular curve of the image equation (2) and the necessary condition for the existence of such a singularity is that the Jacobian determinant J is Zero.

$$J = \frac{\partial x'}{\partial r}\frac{\partial y'}{\partial \theta} - \frac{\partial x'}{\partial \theta}\frac{\partial y'}{\partial r} = 0. \qquad (3)$$

The vector $\mathbf{r}_m$ is the projection of r onto the image plane and can easily be determined. The light beam impinges under a small angle of incidence on the specimen. It is partly refracted through the thickness, then partly reflected on the back surface, and again partly refracted through the thickness when emerging from the specimen. This twice refracted and once reflected part of the light ray is absolutely retarded when passing through the specimen according to Maxwell and Neumann's law. The absolute retardation of the light rays depends on the change of the refractive index and the thickness variation of the plate. The emerging wave front satisfies the Eikonal relation [7] according to which the gradient of the geometric wave front S is constant. The vector $\mathbf{w}$ is given as

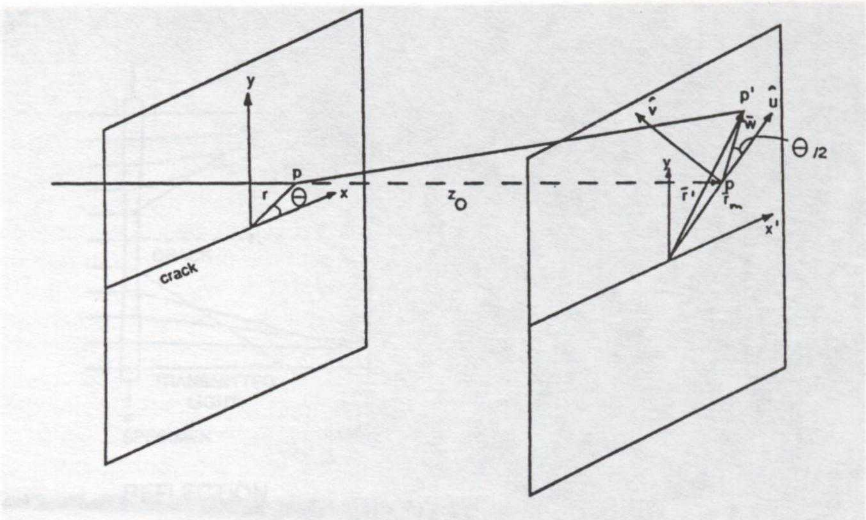

Fig. 7. Shadow optical image geometry.

$$\mathbf{w}(r, \theta) = Z_0 \mathbf{grad}\, \Delta s(r, \theta). \quad (4)$$

where Δs is the change of the optical path length caused by the specimen and Z_0 is the distance from the model to the screen. The path length change Δs is correlated to the stresses $\sigma(r, \theta)$ by the basic elasto-optical equations. The change in the optical path is given by [8]

$$\Delta s = c_r d(\sigma_1 + \sigma_2). \quad (5a)$$

$$\Delta s = c_f d(\sigma_1 + \sigma_2). \quad (5b)$$

$$\Delta s = c_t d(\sigma_1 + \sigma_2). \quad (5c)$$

where c_f *and* c_r are the reflected from the front face and rear face stress optical constants, respectively; c_t is the transmitted caustic stress optical constant; d is the optical path thickness (specimen thickness) σ_1 and σ_2 are the principal stresses.

For mode I SIF, the sum of the principal stresses is

$$\sigma_1 + \sigma_2 = \sigma_x + \sigma_y = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}. \quad (6)$$

Introducing Equation (5) into Equation (4) yields

$$\mathbf{w} = Z_0 dc\, \mathbf{grad}(\sigma_1 + \sigma_2). \quad (7)$$

Where C is C_f for the caustic reflected from the front face, C_r for the reflected caustic from the rear face, and C_t for the transmitted caustic.

If the sum of the principal stresses, Equation (6), is introduced in Equation (7), the deviation vector $\mathbf{w}$ in cartesian coordinate $(\mathbf{u}, \mathbf{v})$ shown in Fig. 7 is given by

$$\mathbf{w} = \delta r^{-3/2} \left(K_I \cos\frac{\theta}{2}\, \mathbf{u} + K_I \sin\frac{\theta}{2}\, \mathbf{v} \right). \quad (8)$$

$$\text{where } \delta = \frac{Z_0 dc}{(2\pi)^{1/2}} \quad (9)$$

From equations (2a) and (8) the vector $\mathbf{r}'$ in cartesian coordinates (x, y) is

$$\mathbf{r}' = x\mathbf{i} + y\mathbf{j} = \left(r\cos\theta + \delta r^{-3/2} K_I \cos\frac{3\theta}{2} \right)\mathbf{i}$$

$$+ \left(r\sin\theta + \delta r^{-3/2} K_I \sin\frac{3\theta}{2} \right)\mathbf{j}.$$

The evaluation of equation (3) gives

$$r = r_0 = \left(\frac{3\delta}{2\lambda} \right)^{2/5} K_I^{2/5}. \quad (11)$$

Equation (11) indicates that the constrained zone around the crack tip subjected to mode I deformation is a circle of radius r_0 (the initial curve) and is a function of K_I, the distance Z_0, and the model thickness d. Using equation (11) in equation (10), then the image equation becomes

$$x' = \lambda r_0 \left(\cos\theta + \frac{2}{3} \cos\frac{3\theta}{2} \right). \quad (12a)$$

$$y' = \lambda r_0 \left(\sin\theta + \frac{2}{3} \sin\frac{3\theta}{2} \right) \quad (12b)$$

When prime (′) means screen coordinates. The angle θ varies beween 0 and 2π for the transmitted caustic and between 0 and 4π for the reflected caustics. The caustic image has the shape shown in Fig. 8. Mode I SIF affects the size of caustic, which is generalised epicycloids as shown in Fig. 8.

If we take into consideration that $z' = x' + iy' = \rho\exp(i\phi)$, then it has been shown that the maximum transverse diameter $D_t\max = 3.17r_0$) [6, 9]. then mode I SIF can be determined from

$$K_I = \frac{0.93375}{Z_0 cd\lambda} \left(\frac{D_t max}{\lambda} \right)^{5/2}. \quad (13)$$

and

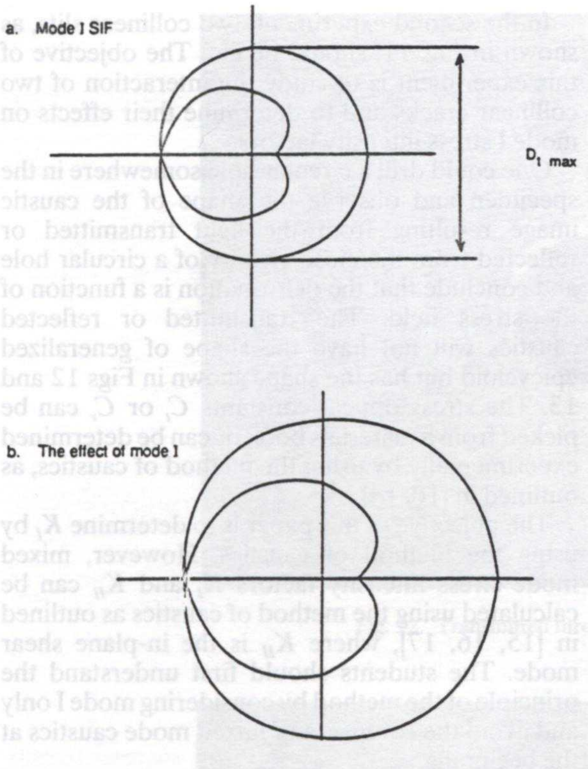

Fig. 8. The shape and size of different generalized epicycloids.

$$\lambda = \frac{Z_i + Z_0}{Z_0} \tag{14}$$

Where λ is the magnification factor and Z_i is the distance between the focus of the impinging light beam and the specimen as shown in Figs. 3 and 4. Thus, with Eq. (13) and the distance $D_t$max measured in experimentally obtained caustic, mode I stress intensity factor K_I can be determined.

Material

In the method of caustics, the material that can be used is polymethyl methacrylate (PMMA, plexiglas). In most experiments, plexiglas has proven to be a suitable material. It has the advantage of being a mechanically and optically isotropic material. Furthermore, it does not present an extensive plastic zone at room temperature even in the close vicinity of the crack tip for sufficiently large loading. Younis and Zachary developed a technique for the determination of stress-optical constants using the method of caustics [10].

Test specimen and model preparation

To ensure a uniform tensile-type load that produces a uniform stress σ, from the authors' experience, models with the dimensions shown in Fig. 9 should be used. We found that the crack lengths must be greater than the model thickness in order to get accurate results. While the exterior geometry was easily machined, achieving proper crack tip conditions was important. For the simple linear approach of the stress distribution around a stationary crack it is sufficient to replace an edge crack with an external slit which is perpendicular to the longitudinal boundary of the model and has a very small radius of curvature. The selected plates were first machined to their final sizes, the slit was then made by means of a metallic disc cutter with a thickness on the order of 0.0236 mm to 0.0314 mm (0.006 in. to 0.008 in.). Since, the root radius of the slit was sufficiently sharp, radius of curvature approaching zero, the slit simulates a real crack edge.

Optical calibration

The screen must always be parallel to the model. The rotation of the screen affects the shape and size of the caustic [11]. The magnification factor can be determined by using the following relation.

$$\lambda = \frac{\textit{any length in the reference plane}}{\textit{corresponding length in the image plane}} \tag{15}$$

However, if the screen is not parallel to the model, an error in the evaluation of magnification factor is obtained. The difference between the calculated magnification factor from Equations (14) and (15) indicates the extent of the errors. The main source of error is that the screen is not parallel to the model and can be easily eliminated.

The three dimensional effects

It has been demonstrated experimentally that the radius r_0 of the initial curve affects the results, and it is necessary to use the value of r_0 larger than a definite one. It has been demonstrated that if r_0 is not large enough in comparison to the thickness of the specimen, the three dimensional effects produce significant errors and r_0 should be at least 0.4

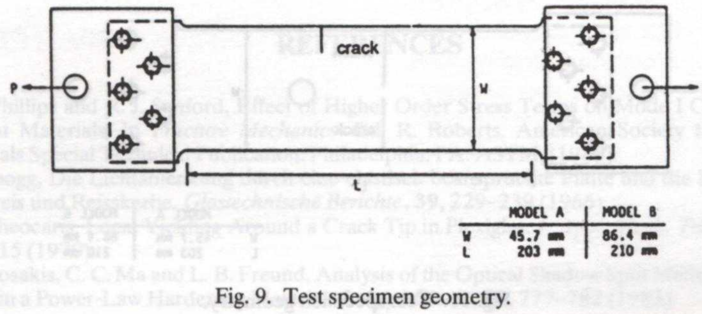

Fig. 9. Test specimen geometry.

times the plate thickness [12]. As mentioned before, r_0 can be calculated from

$$r_0 = \frac{D_t max}{3.17\lambda}. \qquad (16)$$

CONCLUSIONS

It has been established that by using the method of caustics, it is possible to determine mode I stress intensity factor at a crack tip quickly and accurately. The reason for using a laser light beam is that such a beam has a greater intensity than an ordinary light source beam and can be concentrated in the vicinity of the crack tip to produce a clear caustic image. The material that can be used is plexiglas, which makes the method economical and suitable for use in the laboratory. The students should learn how to calculate the size of deformation at a crack tip as well as mode I stress intensity factors. The authors would like to suggest the following experiments. In the first one, two collinear slits of different sizes normal to the boundary should be cut as shown in Fig. 10. From the reflected or transmitted caustic image (or both) one can calculate the initial curve radius (the size of deformation) using

$$r_0 = \frac{D_t max}{3.17\lambda}$$

and mode I stress intensity factor using equation (13). It should be concluded that deformation and K_I are a function of the crack size. In this experiment K_{IB} (K_I at tip B) is greater than K_{IA} (K_I at tip A) and the two values should be compared to the theoretical results available in design books.

In the second experiment, two collinear slits, as shown in Fig. 11, should be cut. The objective of this experiment is to study the interaction of two collinear cracks and to determine their effects on mode I stress intensity factors.

One could drill a circular hole somewhere in the specimen and observe the shape of the caustic image resulting from the light transmitted or reflected from the close vicinity of a circular hole and conclude that the deformation is a function of the stress field. The transmitted or reflected caustics will not have the shape of generalized epicycloid but has the shape shown in Figs 12 and 13. The stress optical constants C_t or C_r can be picked from a materials book or can be determined experimentally by using the method of caustics, as outlined in [10, 14].

The objective of this paper is to determine K_I by using the method of caustics. However, mixed mode stress intensity factors K_I and K_{II} can be calculated using the method of caustics as outlined in [15, 16, 17], where K_{II} is the in-plane shear mode. The students should first understand the principle of the method by considering mode I only and avoid the confusion of mixed mode caustics at the beginning.

The courses that could benefit from this paper are in the field of solid mechanics. Among these are courses that include or are dedicated to, the studies of experimental stress analysis and fracture mechanics. These courses are useful to civil, mechanical, aerospace and materials engineering for example. Although these topics are generally encountered in graduate school, they can be taught to Seniors and in some special cases, to Juniors.

Naturally, the hours of teaching involved depend upon how extensive a treatment of caustics one wishes to offer. Our experience, however, is that

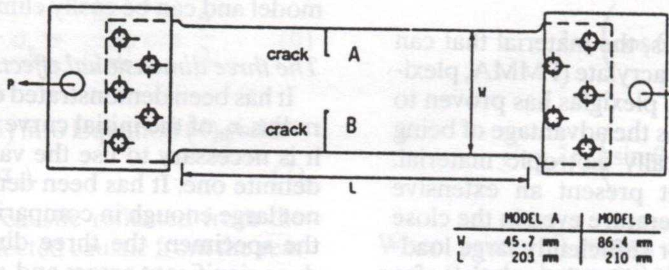

Fig. 10. Test specimen geometry.

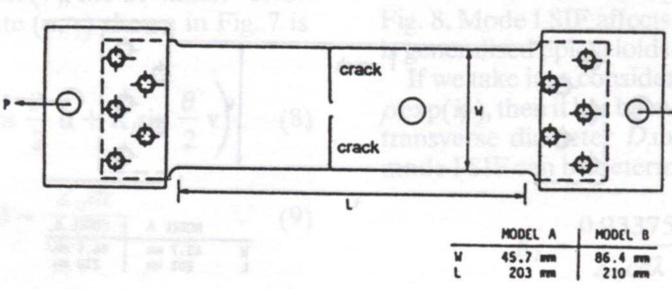

Fig. 11. Test specimen geometry.

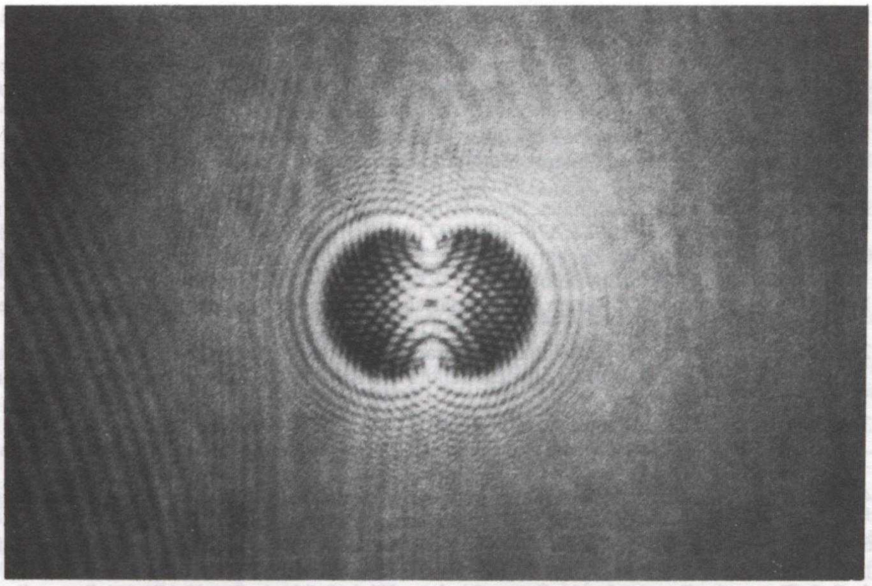

Fig. 12. Transmitted through a central hole caustic pattern.

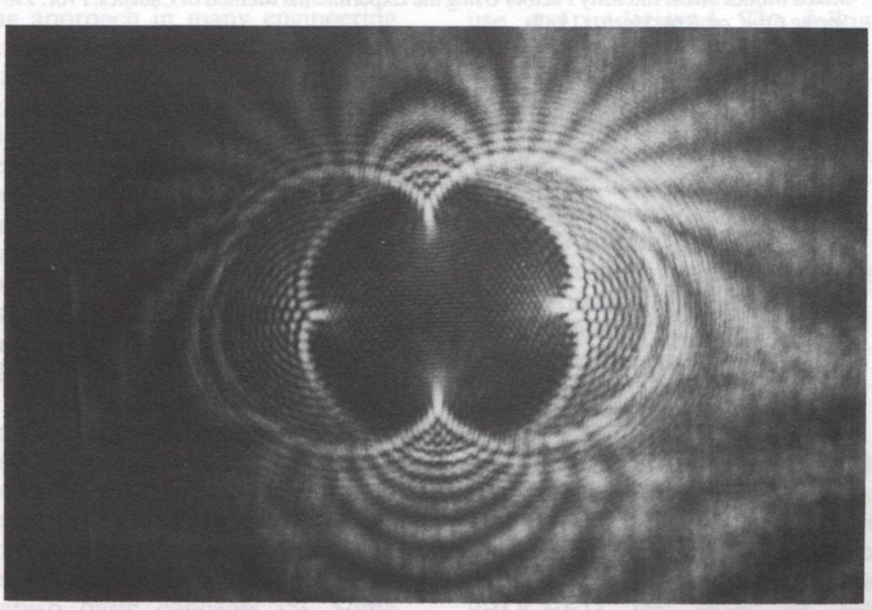

Fig. 13. Reflected from a central hole caustic pattern.

ten (10) hours are sufficient to both introduce the students to the method and to give them enough hands-on experience with it so that they can appreciate its merits as well as its limitations. It must be borne in mind that, for us, the technique is not taught simply for its own sake. Rather, it is presented as a means of demonstrating and quantifying how much the presence of cracks and holes change stress distributions in structural members and machine parts.

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