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Digital Simulation of Kinematic Machinery Systems*

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A technique is presented for modeling prescribed motion machinery systems using a digital computer and a simulation language appropriate to the particular computer available. The technique utilizes loop equations to determine the nonlinear ordinary differential equations describing the motion of the machinery system. The set of differential equations is treated as an initial value problem to determine accelerations, velocities, displacements and reaction forces for a cycle of motion or for whatever time interval is of interest. Utilization of a simulation language for the solution of the equations representing the physical system allows the generation of accurate, reliable numbers for variables of interest with a minimum involvement in the details of the numerical techniques being utilized. An example is presented to illustrate the technique.

The presentation in this paper is such that an instructor who has little or no experience using a digital simulation language to model kinematic systems could use the material presented here to get up to speed and take a class through a series of problems of increasing complexity. The advantage of the digital simulation approach is that the variables of interest are obtained for the full cycle of motion of the device being modeled rather than just an isolated value of the input motion variable.

INTRODUCTION

CLASSICAL techniques for the analytical and graphical analysis of kinematic systems, i.e. prescribed motion problems, utilize a progression which involves the sequential solution of the position, velocity and acceleration problems. These techniques have generally resulted in the determination of the unknown kinematic variables at a single instant in time. If one is interested in the kinematic variables over a full cycle of operation, the analysis, although tedious, can be performed at successive positions of the input members. These techniques are detailed in the standard kinematics textbooks [1].

An alternative approach is to represent the device by a set of differential equations and integrate these equations over a cycle of operation or some other appropriate time interval. The differential equations which describe machinery systems tend to be highly nonlinear and coupled in the highest order derivatives which precludes a nice, tidy, closed-form solution. For this reason it is necessary to resort to a numerical integration procedure on a digital computer to generate solutions to the system of differential equations which result once a mathematical model of a machinery system has been developed. Fortunately, software packages are available which reduce the simulation of systems described by sets of ordinary differential equations to a rather low level of difficulty. Some of those typically available and suitable to this task are ACSL, CSMP and TUTSIM.

The use of digital computers for the simulation of machinery systems was initiated in the late sixties and early seventies. Paul [2] presents an overview of the development of several of the early computer-oriented, machinery simulation programs. The development efforts resulting in IMP [3], DRAM [4], MEDUSA [5], DYMAC [6] and ADAMS [7] are acknowledged. These programs offer the user an easy means of determining the forces and motions in typical machinery systems if the system falls within the scope of problems covered by the program.

The approach adopted in this paper is to have the instructor and/or the student develop a set of appropriate differential equations to represent the system being simulated. The base for the equations are the loop displacement constraint equations appropriate to the physical system being modeled. In the context of this paper the equations are solved and integrated using the digital simulation language ACSL [8], although any of the other general-purpose simulation languages could be employed. The technique can yield any force, torque, or motion information about the system being simulated. In addition, the technique is applicable to kinematic problems (prescribed motion) and to dynamic problems (prescribed force). This paper details the development and implementation of the appropriate equations for kinematic systems while a sequel paper will revise the methodology to a form appropriate for dynamic systems which are often referred to as prescribed force problems.

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The approach adopted here has similarities to the work of Paul [9] and Timm [10]. In [9] the emphasis is on integrating velocities to obtain displacements. In the present paper the accelerations are utilized to maintain dimensional compatibility with the force equations and for ease of reaction force calculations. The accelerations are integrated twice to obtain velocity and displacement information. Additionally, the problems associated with linking subroutines together, with one exception, are removed by the use of a simulation language such as ACSL.

The digital simulation approach developed by Timm [10] was an outgrowth of his earlier work utilizing analog computers to simulate machinery systems [11] and included the use of rate resolvers for angular variables and explicit equations for the motion variables. Experience has indicated that rate resolvers are unnecessary and accurate simulations are obtained in less time if the motion variables are not solved for explicitly. Thus, neither rate resolvers nor explicit equations are used in the present work.

SIMULATION LANGUAGE—GENERAL CONSIDERATIONS

The specific simulation language which will be utilized for the example problem is ACSL. This language is quite similar to other languages such as CSMP, CSSL, TUTSIM, and SL-1. The objective of each of these software packages is to simplify the task of modeling physical systems which are described by sets of ordinary, usually nonlinear, differential equations.

The most powerful feature of the simulation packages is the integration algorithm. These are 'black boxes' which the user calls to solve systems of differential equations without being concerned about the actual details involved in numerical integration. The algorithm can be represented diagrammatically as shown in the top two blocks of Fig. 1.

Suppose the simulation involves a second order differential equation in an angular variable θ. The input variables to the integrators are $\alpha(\ddot{\theta})$ and $\omega(\dot{\theta}^t)$ and initial conditions on θ and ω. The initial conditions are used once at the start of the simulation which is usually at time $t = 0$. When the values of $\ddot{\theta}$ and $\dot{\theta}^t$ which are input to the integrators are operated upon by the numerical integration algorithm the results are output quantities $\dot{\theta}^t$ and θ. If the input $\ddot{\theta}$ and $\dot{\theta}^t$ are instantaneous values at time t, the output quantities $\dot{\theta}^t$ and θ are the values for these variables at $t + \Delta t$, a short time later than t.

The differential equations which represent a mathematical model of a machinery system are typically coupled in the higher order terms and are usually so nonlinear that there is no way to uncouple them which is valid for any reasonable time interval or range of motions. Thus, in addition to the integration problem a function evaluation must be performed. The function evaluation can also be represented by a 'black box', and is indicated schematically in the bottom block of Fig. 1.

In terms of inputs and outputs the function evaluation process can be described as follows. Given all the position and velocity information at a particular instant of time, calculate the corresponding accelerations. Notice that the function evaluation process takes place at a particular instant in time. The integration process advances time and provides new velocity and position information. A problem involving a single angular coordinate described by a second order differential equation could conceptually be solved by the entire diagram of Fig. 1.

AN ACCELERATION-BASED SIMULATION

As an example of the technique discussed consider a kinematic model of the slider-crank device of Fig. 2. The model can be easily generated by writing the constraint imposed connections of the linkage in the form of a kinematic loop equation or

$$\bar{R}_2 + \bar{R}_3 + \bar{S} = 0 \qquad (1)$$

If this expression is differentiated with respect to time and angles are defined positively counterc-

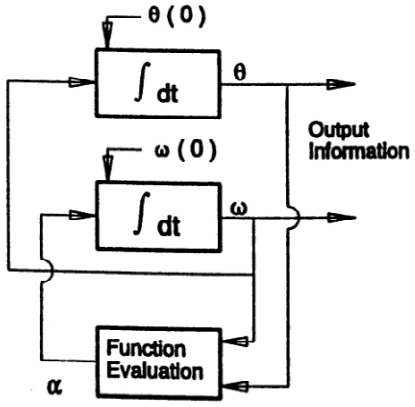

Fig. 1. Schematic representation of a simple simulation.

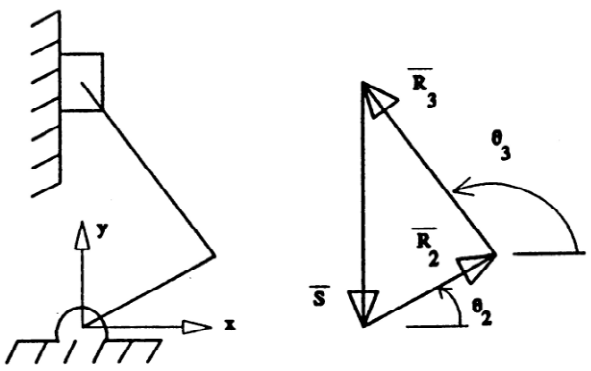

Fig. 2. Schematic and vector equivalent of a slider-crank device.

lockwise from the horizontal (x) axis, the vector velocity constraint equation is

$$\bar{\omega}_2 \times \bar{R}_2 + \bar{\omega}_3 \times \bar{R}_3 + \dot{S}\hat{S} = 0 \tag{2}$$

This equation could be written in terms of the scalar components which could be solved by the simulation program for velocity and displacement information. However, if force and/or acceleration information is desired it is appropriate to use a higher order model which is given by further differentiating relation (2) with respect to time to yield

$$\bar{\omega}_2 \times (\bar{\omega}_2 \times \bar{R}_2) + \bar{\alpha}_2 \times \bar{R}_2 + \bar{\omega}_3 \times (\bar{\omega}_3 \times \bar{R}_3) + \bar{\alpha}_3 \times \bar{R}_3 + \ddot{S}\hat{S} = 0. \tag{3}$$

In terms of scalar components this equation yields for the x component

$$-R_2\omega_2^2\cos\theta_2 - R_2\alpha_2\sin\theta_2 - R_3\omega_3^2\cos\theta_3 - R_3\alpha_3\sin\theta_3 = 0 \tag{4}$$

and for the y component

$$-R_2\omega_2^2\sin\theta_2 + R_2\alpha_2\cos\theta_2 - R_3\omega_3^2\sin\theta_3 + R_3\alpha_3\cos\theta_3 - \ddot{S} = 0 \tag{5}$$

Solving expression (4) for α_3 yields

$$\alpha_3 = [R_2\omega_2^2\cos\theta_2 + R_2\alpha_2\sin\theta_2 + R_3\omega_3^2\cos\theta_3]/(-R_3\sin\theta_3) \tag{6}$$

and solving (5) for $\ddot{S}$ gives

$$\ddot{S} = R_2\omega_2^2\sin\theta_2 - R_2\alpha_2\sin\theta_2 + R_3\omega_3^2\sin\theta_3 - R_3\alpha_3\cos\theta_3 \tag{7}$$

Expressions (6) and (7) can be integrated to yield $\omega_3(t)$ and $\dot{S}(t)$ and again to give $\theta_3(t)$ and $S(t)$. For the kinematic model it is assumed that the crank variables θ_2, ω_2 and α_2 are known functions of time.

ACCELERATIONS OF MASS CENTERS

Once the angular acceleration of the connecting rod and the piston acceleration have been determined, sufficient information is available to calculate the linear acceleration of each mass center. Let vectors $\bar{R}_2'$, $\bar{R}_3'$, and $-\bar{S}$ locate the mass centers of parts 2, 3, and 4 as indicated in Fig. 3. The mass center accelerations of the crankshaft, connecting rod and piston are determined by appropriate differentiations.

$$\frac{d^2}{dt^2}(\bar{R}_2') = \bar{\omega}_2 \times (\bar{\omega}_2 \times \bar{R}_2') + \bar{\alpha}_2 \times \bar{R}_2' \tag{8}$$

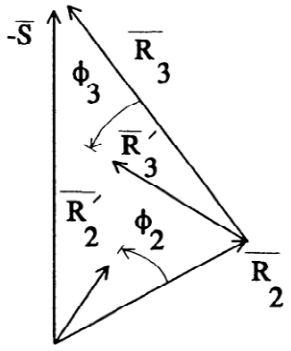

Fig. 3. Mass center locations.

$$\frac{d^2}{dt^2}(\bar{R}_2 + \bar{R}_3') = \bar{\omega}_2 \times (\bar{\omega}_2 \times \bar{R}_2) + \bar{\alpha}_2 \times \bar{R}_2 + \bar{\omega}_3 \times (\bar{\omega}_3 \times \bar{R}_3') + \bar{\alpha}_3 \times \bar{R}_3' \tag{9}$$

$$\frac{d^2}{dt^2}(-\bar{S}) = -\ddot{S}\hat{j} \tag{10}$$

The rectangular components of the vector equations (8) and (9) are given below. The variable $ACM2I$ is interpreted as the acceleration of the center of mass of part 2 in the x direction. The other variables are similarly interpreted.

$$\begin{aligned}
ACM2I = {} & -R_2'\omega_2^2\cos(\theta_2 + \phi_2) - R_2'\alpha_2\sin(\theta_2 + \phi_2) \\
ACM2J = {} & -R_2'\omega_2^2\sin(\theta_2 + \phi_2) + R_2'\alpha_2\cos(\theta_2 + \phi_2) \\
ACM3I = {} & -R_2\omega_2^2\cos(\theta_2) - R_2\alpha_2\sin(\theta_2) - R_3'\omega_3^2\cos(\theta_3 + \phi_3) - R_3'\alpha_3\sin(\theta_3 + \phi_3) \\
ACM3J = {} & -R_2\omega_2^2\sin(\theta_2) + R_2\alpha_2\cos(\theta_2) - R_3'\omega_3^2\sin(\theta_3 + \phi_3) + R_3'\alpha_3\cos(\theta_3 + \phi_3)
\end{aligned}$$

The model of the slider-crank which uses the acceleration constraint relations can now be modified to include the calculation of mass center accelerations.

These equations are completely general. For a specific example, the mass centers will be located midway along the connecting rod, at the wrist pin connection for the slider, and the crankshaft will be assumed balanced.

REACTION FORCE ANALYSIS

Adopt the notation that $\bar{F}_{i,j}$ is the force exerted on link i by link j. The reaction forces acting on the moving parts of the system are shown in Fig. 4. The perspective is adopted that an engine is being modeled so that $\bar{F}_{ext}$ is the agent of driving force and $\bar{T}_{2,1}$ is a load torque, i.e. the torque $\bar{T}_{2,1}$ is the

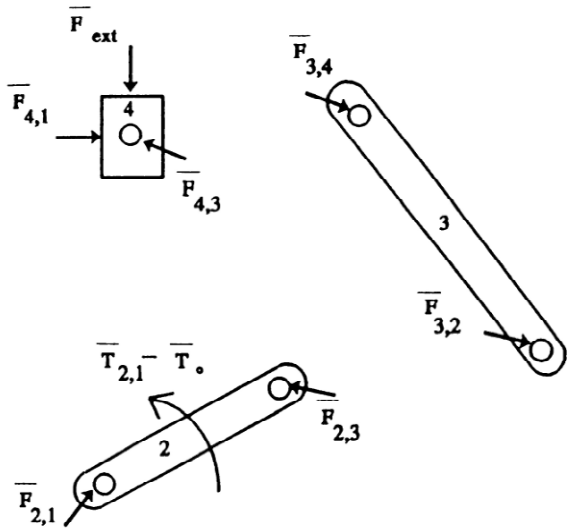

Fig. 4. Applied and reaction forces acting on the slider crank.

torque exerted on 2 by whatever is being driven by the engine. For a kinematic model both of these effects are known.

The magnitudes of the $\bar{F}_{i,j}$ are needed in order to select bearings for the revolute connections and perform deflection and stress analyses on the links. The variation of $\bar{T}_{2,1}$ over a cycle operation is an indication of whether or not a flywheel is necessary. There could be any number of applied forces acting; some load characteristic acting against the rotation of link 2, springs or dampers attached between two moving points or between moving points and ground, or the force of gravity. None of these effects will change the approach to reaction force analysis since they are position- or velocity-dependent effects. These effects only increase the detail of the model. For a crankshaft rotating with constant angular velocity the load effects will be lumped into a constant magnitude viscous plus a square-law drag torque acting against the rotation of the crankshaft. With this load effect the reaction forces will be determined. Newton's second law, in the form of d'Alembert's principle, for each of the moving parts results in:

$$\bar{F}_{\text{ext}} + \bar{F}_{4,1} + \bar{F}_{4,3} - m_4\bar{a}_4 = 0 \tag{11}$$

$$\bar{F}_{3,4} + \bar{F}_{3,2} - m_3\bar{a}_3 = 0 \tag{12}$$

$$\bar{F}_{2,3} + \bar{F}_{2,1} - m_2\bar{a}_2 = 0 \tag{13}$$

$$\bar{R}_2 \times \bar{F}_{2,3} + T_{2,1}\hat{k} - \bar{R}'_2 \times m_2\bar{a}_2 - I_2\bar{\alpha}_2 - T_0\hat{k} = 0 \tag{14}$$

$$\bar{R}_3 \times \bar{F}_{3,4} - \bar{R}'_3 \times m_3\bar{a}_3 - I_3\bar{\alpha}_3 = 0 \tag{15}$$

The vectors $\bar{R}'_2$ and $\bar{R}'_3$ locate the mass centers of parts 2 and 3 from the points of application of $\bar{F}_{2,1}$ and $\bar{F}_{3,2}$ respectively. The set of scalar equations equivalent to the above, after substituting $-\bar{F}_{2,3}$ for $\bar{F}_{3,2}$ and $-\bar{F}_{3,4}$ for $\bar{F}_{4,3}$, is:

$$F_{4,1x} - F_{3,4x} = 0 \tag{16}$$

$$-F_{3,4y} = m_4a_{4y} + F_{\text{ext}} \tag{17}$$

$$F_{3,4x} - F_{2,3x} = m_3a_{3x} \tag{18}$$

$$F_{3,4y} - F_{2,3y} = m_3a_{3y} \tag{19}$$

$$F_{2,3x} + F_{2,1x} = m_2a_{2x} \tag{20}$$

$$F_{2,3y} + F_{2,1y} = m_2a_{2y} \tag{21}$$

$$\begin{aligned} R_2\cos(\theta_2)F_{2,3y} - R_2\sin(\theta_2)F_{2,3x} + T_{2,1} &= I_2\alpha_2 \\ &+ R'_2\cos(\theta_2)m_2a_{2y} \\ &- R'_2\sin(\theta_2)m_2a_{2x} + T_0 \end{aligned} \tag{22}$$

$$\begin{aligned} R_3\cos(\theta_3)F_{3,4y} - R_3\sin(\theta_3)F_{3,4x} &= I_3\alpha_3 \\ &+ R'_3\cos(\theta_3)m_3a_{3y} \\ &- R'_3\sin(\theta_3)m_3a_{3x} \end{aligned} \tag{23}$$

Equation (16) does not include the applied gas force F_{ext} because this force has a y component only. Equation (17) does not contain $F_{4,1}$ because the reaction of the cylinder on the piston is normal to the direction of piston motion for a well lubricated system. In writing equations (22) and (23) it has been assumed that the mass centers of parts 2 and 3 are located along vectors $\bar{R}_2$ and $\bar{R}_3$ by the vectors $\bar{R}'_2$ and $\bar{R}'_3$. This is not necessary, however it simplifies the example. If this is not representative of the actual device the effect of being offset from vectors $\bar{R}_2$ and $\bar{R}_3$ could easily be included.

Equations (16)–(23) involve eight scalar unknowns. Thus, the eight equations could be solved for the reaction forces at each integration step using a simultaneous linear equation solver. Another procedure results when it is recognized that the reaction force equations are already uncoupled if the calculations proceed in a specific order. At any value of time the value $F_{3,4y}$ can be calculated from Equation (17) once the acceleration of the piston, $\ddot{S}$, is known. Follow this with the calculation of $F_{2,3y}$ from Equation (19), then $F_{2,1y}$ from Equation (21). After these calculations have been performed $F_{3,4x}$ can be determined from (23). $F_{4,1x}$ and $F_{2,3x}$ are then determined from Equations (16) and (18). Equation (20) is then used to calculate $F_{2,1x}$. Finally, the torque $T_{2,1}$, which forces the constant crankshaft angular velocity, is calculated from Equation (22). The external force acting on the piston will be modeled as an air standard Otto cycle with a compression ratio of 8:1.

KINEMATIC SLIDER CRANK RESULTS

The kinematic slider-crank model developed above was implemented in ACSL. The values for physical parameters used are tabulated in Table 1. A copy of the ACSL code is available from the authors on request.

Results for the reaction forces and required driving torque to enforce the assumed constant velocity of the crankshaft are shown in Fig. 5.

When comparing the numerical data used to

Table 1. Parameters for kinematic slider crank simulation

Link lengths	Masses	Mass moments of inertia
$R_2 = 0.0508$ (m)	$M_2 = 17.5$ (kg)	$I_2 = 0.0452$ (kg m^2)
$R_3 = 0.2032$	$M_3 = 87.5$	$I_3 = 0.0113$
	$M_4 = 4.38$	

Drag torque $T_0 = 0.113\ \omega_2 + 0.00113 * \omega_2 * \omega_2$ (Nm)

Gas force applied to piston (air standard Otto cycle—two stroke)
FEXT = 3880.4/(0.2685 − S)**1.4 (compression)
FEXT = 19116.7/(0.2685 − S)**1.4 (expansion)
Driving motion
$\omega_2 = 252.336759$ rad/s (assumed constant)
For convenience $R'_2 = 0.0$ (balanced crankshaft) and $R'_3 = R_3/2.0$
(e.g. midway between bearings)

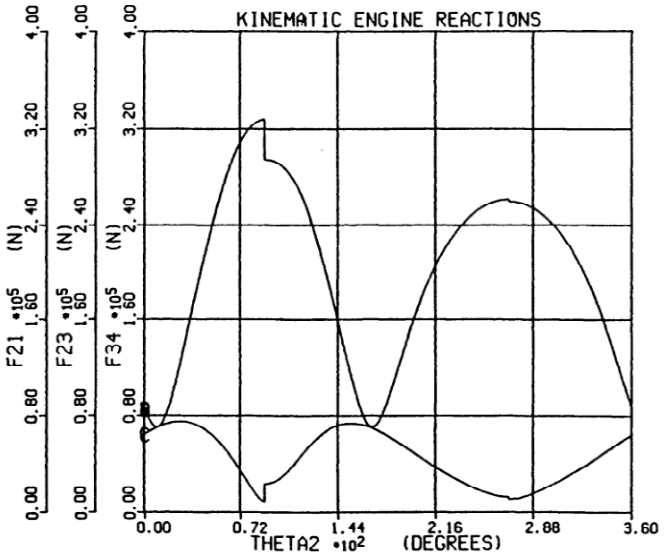

Fig. 5. Plots of the kinematic engine reactions.

generate Fig. 5 for the start of the cycle and the end of the cycle, the force values are identical and the torque value differs in the 6th significant figure.

In Fig. 5, time = 0 corresponds to the crankshaft being horizontal and rotation counterclockwise. Top dead center occurs at THETA2 = 90°, bottom dead center is at THETA2 = 270°. The top curve is the force at the crankshaft main bearings and the crankshaft-connecting rod connection. These reactions are identical since the crankshaft was assumed perfectly balanced. The lower curve is the reaction at the piston and connecting rod bearing.

CONCLUSIONS

A technique has been presented which reduces the simulation of kinematic systems to a rather straightforward process of implementing loop constraint equations and d'Alembert's principle to obtain any desired motion or force information about the system being modeled.

The approach parallels that of Timm [10] with two major exceptions. Rate resolvers are not utilized and no attempt is made to reduce the number of motion or force equations through extensive algebraic manipulation. The import and advantage of avoiding these algebraic manipulations is more evident in a sequel paper where prescribed force systems are considered.

The technique described has advantages from both the student and instructor viewpoints. It reinforces Newtonian and d'Alembertian concepts from an elementary dynamics course without recourse to any advanced concepts such as Lagrangian mechanics. It allows the student to simulate physical devices through a complete motion cycle rather than focusing on a single position utilizing hand calculations or graphical techniques.

This approach also allows the instructor to approach kinematic motion and force analysis from a general point of view by treating the system as a coupled set of nonlinear differential equations which require digital computation for their solution. The students solve nonlinear differential

equations in a context that lends itself quite well to checking the solution by classical kinematic analysis techniques.

SUMMARY

This paper details the use of a continuous system simulation language for modeling kinematic machinery systems. Rather than the classical approach of a sequential position, velocity and acceleration analysis the authors use a force-acceleration model and treat the problem as an initial value, differential equation problem. In an undergraduate education environment this approach reinforces the students' basic mechanics background while they experience a solution methodology for nonlinear differential equations where the numerical results can be verified by classical techniques involving hand calculations or graphical methods.

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