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Integrating Uncertainty Analysis Concepts into Undergraduate Laboratory Courses*

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Uncertainty analysis concepts have been integrated into the undergraduate measurements laboratory sequence in Mechanical Engineering at Mississippi State University. One of the main objectives of this three-course sequence is to present the concepts and techniques that are necessary for the students to find solutions to problems using an experimental approach. The students first learn about transducers and data acquisition and are then introduced to the concepts of experiment planning, design, debugging, execution, and data analysis. Throughout this process they perform experiments of increasing difficulty. This paper describes the key aspects of how uncertainty analysis techniques are used in this laboratory sequence and provides examples of the applications.

INTRODUCTION

ERRORS are present in every experimental measurement, and uncertainty analysis allows examination of the effects of these errors in all phases of experimental programs. At Mississippi State University we have integrated uncertainty analysis concepts into our Mechanical Engineering laboratory sequence. This allows us to present a logical approach to experimentation through the application of uncertainty analysis in the planning, design, construction, debugging, execution, data analysis, and reporting phases of experimental programs.

The three-semester measurements laboratory sequence consists of three one-credit-hour courses in which the students meet for 3 hours one afternoon each week for 14 weeks. In the first course the students are introduced to the basics of experimentation including transducers, signal conditioners, and analog, digital, and microcomputer based data acquisition. The students become familiar with most of the types of typical mechanical engineering measurements. An introductory level measurement techniques text [1] is used in this course.

In the second course the students are introduced to the concepts and applications of uncertainty analysis. This introduction is done in the context of planning, designing, and executing four experiments and analyzing and reporting the results. The course material is arranged so that the students' degree of understanding of uncertainty analysis grows as the semester progresses. The text used for this course is *Experimentation and Uncertainty Analysis for Engineers* by Coleman and Steele [2].

In the initial portion of this second course the topics presented are the philosophy of problem solving via experimentation; the meanings of such concepts as experimental accuracy, precision (random) errors and bias (fixed) errors, and repetition and replication; and the use of uncertainties as estimates of errors. The basics of the Gaussian distribution of random errors, confidence intervals, and the Student's t-distribution are studied, and an experiment is conducted in which the conditions are set and readings are made by each of the students separately to illustrate the idea of precision errors.

In the next segment of this second course, the concepts of general uncertainty analysis are developed and used in the context of planning an experiment. The students learn how to determine the effects of measurement errors on the result of the experiment and how to consider options for modifying the experimental program based on the results of their uncertainty analyses. They then plan and conduct the second experiment in the course.

Following this planning phase, detailed uncertainty analysis is studied as the technique for correctly handling bias and precision errors and for determining their effects on the experimental uncertainty. The students do a detailed uncertainty analysis of an experiment, conduct it, and then determine the experimental result along with its uncertainty.

The last part of the second course deals with experiment operation in the debugging and execution phases. The fourth experiment illustrates the concepts of data analysis and reporting for the situation in which a curve fit for the experimental result over a certain range of variables is desired

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along with a statement of uncertainty associated with the equation obtained.

The third course in the three-course sequence is a capstone design laboratory in which the students apply the principles of the previous two courses to the planning, design, construction, operation, and analysis of the results of several experiments. They are divided into teams, and each team is given a different assignment for each experiment. The goal of this course , and of the three-course sequence, is to give the students the tools necessary to handle experimental programs properly, that is, to find appropriate solutions to problems using an experimental approach.

Now that we have presented an overview of the three-course laboratory sequence at Mississippi State University, some of the specific topics covered in the second course will be discussed. In the following sections of this paper, details on the uncertainty analysis concepts covered in this course are described.

UNCERTAINTY ANALYSIS

General uncertainty analysis

What uncertainty should be associated with experimental results that are determined by combining several measured variables using a data reduction equation? The measurements of the variables (temperature, pressure, velocity, etc.) have uncertainties associated with them, and the values of the material properties which we obtain from reference sources also have uncertainties. How do the uncertainties in the individual variables propagate through a data reduction equation into a result? This is a key question in experimentation, and its answer is found using uncertainty analysis.

In the planning of an experimental program, the approach we use considers only the overall measurement uncertainties and not the details of the bias and precision components of those uncertainties. We call this approach *general uncertainty analysis*. It makes sense to consider only the overall uncertainty in each measured variable at this stage rather than worry about which part of the uncertainty will be due to bias and which part will be due to precision errors. In what we call the planning phase, we try to identify which experiment or experiments will give us a chance to answer the questions we have. In general, the particular equipment and instruments will not have been chosen—whether a temperature will be measured with a thermocouple, thermometer, thermistor, RTD, or optical pyrometer, for instance, is yet to be decided.

Consider a general case in which an experimental result r, is a function of J variables X_i

$$r = r(X_1, X_2, \ldots, X_J) \quad (1)$$

Equation (1) is the data reduction equation used for determining r from the measured values of the variables X_i. Then the manner in which the measurement uncertainties propagate into an uncertainty in the result is given by the expression developed by Kline and McClintock [3],

$$U_r = \left[\left(\frac{\partial r}{\partial X_1} U_{X_1}\right)^2 + \left(\frac{\partial r}{\partial X_2} U_{X_2}\right)^2 + \ldots + \left(\frac{\partial r}{\partial X_J} U_{X_J}\right)^2\right]^{1/2} \quad (2)$$

where the U_{X_i} are the uncertainties in the measured variables X_i. All of the uncertainties (U_{X_i}) in Equation (2) represent the intervals around the measured values X_i within which the true values of X_i are expected to lie. These uncertainties are all expressed with the same degree of confidence, usually 95% confidence (or 20–1 odds).

A primary point which is made to the students is that a general uncertainty analysis should be used in the very initial stages of an experimental program. The information and insight gained are far out of proportion to the small amount of time the analysis takes, and parametric analysis using a range of assumed values for the uncertainties is perfectly acceptable.

Example. Consider the following example of the use of general uncertainty analysis in the planning phase of an experimental program. A company needs to determine the solid particulate concentration in the exhaust gases from one of its process units. Monitoring the concentration will allow immediate recognition of unexpected and undesired changes in the operating condition of the unit. The measurement is also necessary to determine and monitor compliance with air pollution regulations. The measurement needs to be accurate to within about 5%, with a 10% or greater inaccuracy being unacceptable.

A laser transmissometer system is proposed to measure the projected area concentration of the particulate matter in the stack exhaust. A schematic of the laser transmissometer system is shown in Fig. 1. A laser beam passes through the exhaust gas stream and impinges on a transducer which measures the beam intensity. The physical process is described [4] by the expression

$$T = \frac{I}{I_0} = e^{-CEL} \quad (3)$$

In this expression, I_0 is the intensity of the light beam exiting the laser, I is the transmitted intensity after the beam has passed through the scattering and absorbing medium of thickness L, and T is the fractional transmitted intensity (the transmittance). The projected area concentration C is the projected area of the particulates per unit volume of the medium. The extinction coefficient E is a function of the optical properties of the particulates but has an asymptotic value of 2.0 for the conditions of interest.

If we assume a 1% uncertainty for all the measurements and perform a general uncertainty

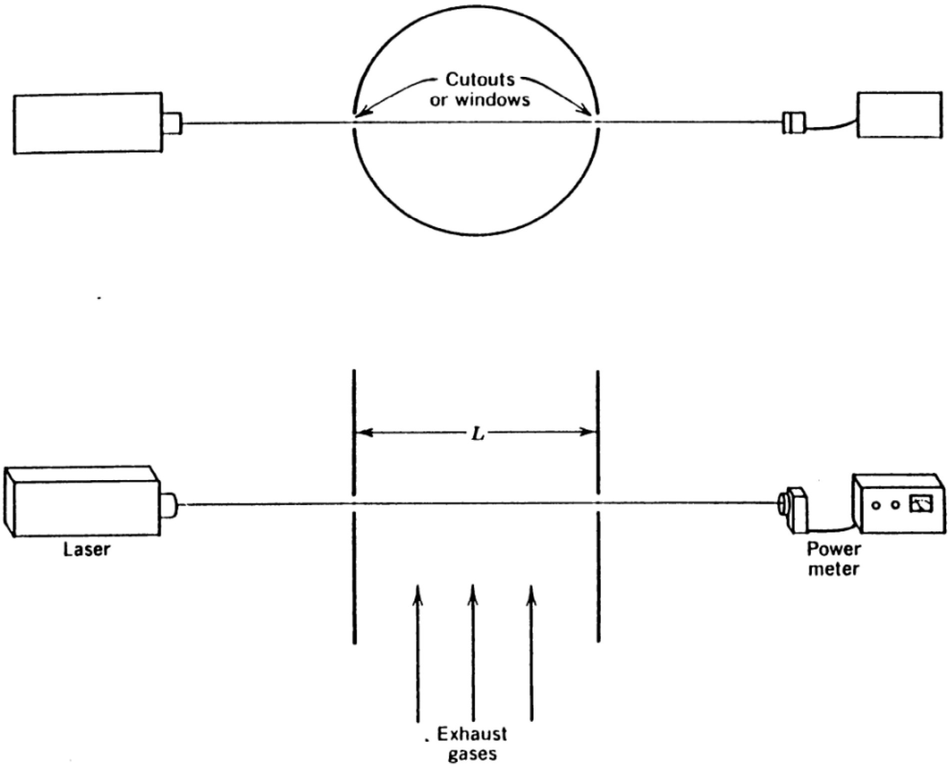

Fig. 1. Schematic of a laser transmissometer system for monitoring particulate concentrations in exhaust gases.

analysis over the range of possible values of transmittance (0 to 1), the results given in Fig. 2 are obtained. The 1% uncertainties that we have assumed may or may not be realistic, but at this point we are merely considering the feasibility of the experiment.

Looking at Fig. 2, we see that the uncertainty in the concentration increases significantly as the transmittance approaches 0.9 or greater. In an actual exhaust, an acceptable transmittance would probably be greater than 0.9. If the acceptable transmittance for operation of the plant were 0.95, then the uncertainty in the particulate concentration measurements would be about 20%. This value is greater than our specified requirement of 5–10% uncertainty. Since we are doing an uncertainty analysis in the planning phase of an experiment, we are free to play 'what if' with almost no restrictions. In fact, playing 'what if' at this stage of an experimental design should be encouraged.

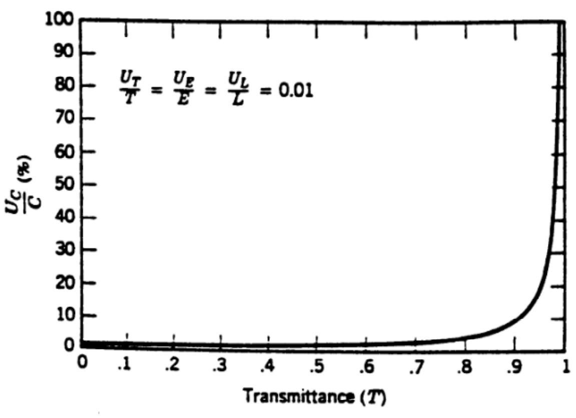

Fig. 2. Uncertainty in the experimental result as a function of the transmittance.

Consider the expression for the transmittance

$$T = e^{-CEL} \tag{3}$$

A decrease in the transmittance would put the operation of our system in a more acceptable uncertainty range as shown in Fig. 2. For a given set of operating conditions, the characteristics of the exhaust flow are fixed and therefore C and E are fixed. The only way to cause a decrease in T, then, is to increase L, the path length of the light beam through the exhaust stream. Rather than recommend that a new, larger diameter stack be constructed (!), we might recommend that two mirrors on adjustable mounts be purchased and used as shown in Fig. 3.

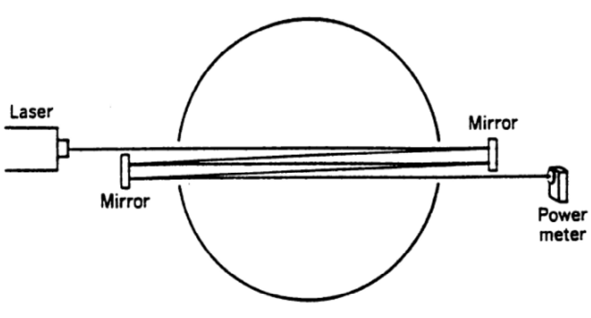

Fig. 3. Multiple pass beam arrangement for the transmissometer.

The two additional mirrors allow multiple passes of the beam through the stack, thus increasing the path length L, decreasing the overall transmittance and decreasing the uncertainty in the measurement of C. For example, consider the case when all measurements are made with 1% uncertainty and the transmittance for one pass of the beam through the stack is 0.95. The effect of additional passes on the transmittance can be calculated from $T(n \text{ passes}) = (0.95)^n$. The behaviour of the uncertainty in the result is shown in Fig. 4 as a function of the number of beam passes through the stack.

The results of the above analysis indicate that the proposed system, with modifications, might be able to meet the requirements. Additional factors, such as the effect of laser output (I_0) variation with time and the probable behaviour of E, should be investigated [5,6] if the implementation of the technique is to be considered further.

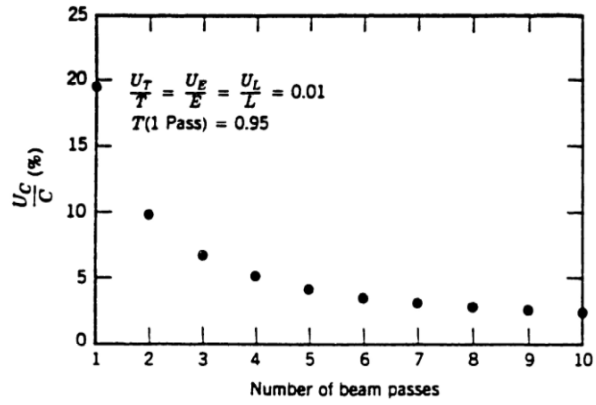

Fig. 4. Uncertainty in the experimental result as a function of the number of beam passes.

Detailed uncertainty analysis

Once the experimental approach has been planned, we turn our attention to details of the specific instruments and techniques to be used. In this *detailed uncertainty analysis*, which is consistent with the approach outlined in the ANSI/ ASME Standard on Measurement Uncertainty [7], the details of the bias and the precision errors in each measured variable are considered, and the propagation of the bias and precision limits into the experimental result are investigated separately.

The primary reason for considering the more complex approach of detailed uncertainty analysis is that it is very useful in the design, construction, debugging, data analysis and reporting phases of an experiment to consider separately the bias and precision components of uncertainty. The bias is a fixed error which can be reduced by calibration but is unaffected by averaging multiple readings. However, the precision error is a variable error which can be reduced by averaging multiple readings. This differing behaviour of the two components of the uncertainty makes it desirable and necessary to consider the components separately.

The situation which we wish to analyze is illustrated in Fig. 5, which shows a flow diagram of the propagation of errors into an experimental result. Each of the measurement systems which is used to measure the value of an individual variable X_i is influenced by a large number of elemental error sources. The effects of these elemental errors are manifested as a bias error (estimated by the bias limit B_i) and a precision error (estimated by the

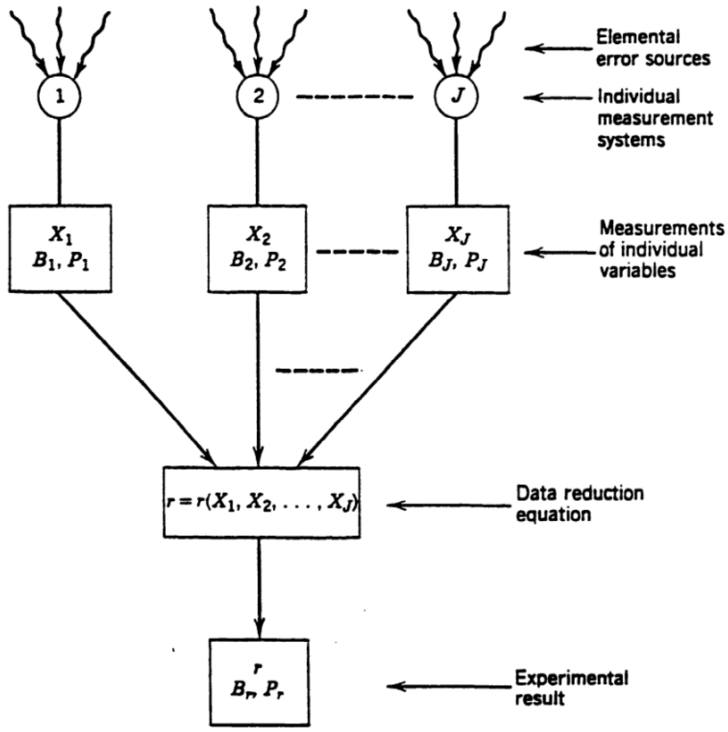

Fig. 5. Propagation of errors into an experimental result.

precision limit P_i) in the measured value of the variable. These errors in the measured values then propagate through the data reduction equation and cause the bias and precision errors in the experimental result, which are estimated by the bias limit B_r and the precision limit P_r.

The procedure in detailed uncertainty analysis is to investigate the contributions of the elemental error sources, obtain estimates of the bias and precision limits for each measured variable, and use the appropriate uncertainty analysis expressions (developed in detail in [2]) to obtain values for the bias limit B_r and precision limit P_r of the experimental result. The uncertainty in the result is then expressed by combining these two error components in one of two ways

$$U_{r_{RSS}} = [B_r^2 + P_r^2]^{1/2} \quad (4)$$

or

$$U_{r_{ADD}} = B_r + P_r \quad (5)$$

The root–sum–square uncertainty, $U_{r_{RSS}}$, results in approximately 95% coverage of the true value. The additive uncertainty, $U_{r_{ADD}}$, results in approximately 99% coverage when the bias and precision contributions are of the same order and 95% coverage when one is negligible relative to the other [7]. These statements assume that the bias limit estimates are all made at 95% confidence (20 to 1 odds) and that the Student's t value in the precision limit ($P = tS$) has been chosen for a 95% confidence level (where S is the precision index). The details of determining and handling the bias and precision errors and of handling the situation where bias errors are correlated are beyond the scope of this paper, but are discussed in considerable detail in [2].

Example. Consider the following example of the use of detailed uncertainty analysis in the design and execution phases of an experimental program. A mining company wishes to determine the heating value of some lignite coal in which it has an interest. They want to know the uncertainty associated with measuring the heating value of a single sample of the material.

The heating value will be determined with an oxygen bomb calorimeter. This device is a standard, commercially available system for making this type of determination. A measured sample of the fuel is enclosed in the metal pressure vessel which is then charged with oxygen. This bomb is then placed in a bucket containing 2000 g of water and the fuel is ignited by means of an electrical fuse. The heat release from the burning fuel is absorbed by the metal bomb and the water. By measuring the temperature rise ($T_2 - T_1$) of the water, the heating value of the fuel can be determined from an energy balance on the calorimeter which gives

$$H = \frac{(T_2 - T_1)C - e}{M} \quad (6)$$

where H is the heating value (cal/g), T is temperature (K), C is the calibrated energy equivalent of the calorimeter (cal/K), e is the energy release from the burned fuse wire (cal), and M is the mass of the fuel sample (g). Note that this is the higher heating value since the vapor formed is condensed to liquid.

In performing a detailed uncertainty analysis for this experiment it is necessary to first obtain estimates of the bias and precision limits for the variables in Equation (6). Using information from previous experiments, calibration data, and uncertainty techniques, appropriate values are obtained and presented in Table 1. Data from a typical test are also given in the table.

Table 1. Estimates of the bias and precision limits for variables in heating value determination and typical test data

Variable	Bias limit	Precision limit	Test value
T_1	±0.1 K	±0.029 K	297.4 K
T_2	±0.1 K	±0.029 K	299.2 K
C	±14.8 cal/K	0	2493 cal/K
e	0	0.7 cal	12 cal
M	0	±0.00028 g	1.0043 g

Using these bias and precision limits in the detailed uncertainty analysis techniques, it is found that the bias limit for the heating value is

$$B_H = \pm 27 \text{ cal/g} \quad (7)$$

and the precision limit is

$$P_H = \pm 102 \text{ cal/g} \quad (8)$$

The uncertainty (with 95% coverage) is then found using Equation (4) as

$$U_H = [(27)^2 + (102)^2]^{1/2} = \pm 106 \text{ cal/g} \quad (9)$$

This uncertainty represents the 95% confidence limit estimate of the error associated with making single determinations of the lignite heating value.

The techniques of detailed uncertainty analysis allow us to answer many other questions about the lignite heating value. If the question were what is the heating value of a portion of the total lignite deposit, then data from many tests could be used to determine the effects of the moisture and ash variability in the samples on the overall uncertainty. This uncertainty would obviously be larger than that for a single sample measurement given in Equation (9). We could also predict the probable ranges of heating values in the deposit.

The key point here is that detailed uncertainty analysis allows us to predict where the true value of the experimental result will lie with a certain degree of confidence. Considering the bias and precision limits of the separate measurements, we are able to determine the controlling contributors in the overall experimental error. In the design phase of the experiment, we can choose to recalibrate or change instruments if we find that the uncertainty of the result will be too large.

RUNNING EXPERIMENTS AND REPORTING DATA

After proceeding through the planning and design phases of an experiment, choosing the instrumentation, and building up the experimental apparatus, typically a debugging and qualification phase is necessary. In this phase, we try to determine and fix the unanticipated problems which occur and to reach the point where we feel confident that the experiment is performing as anticipated.

In the debugging phase of the experiment, we are primarily making two checks on the data. One is a comparison of the scatter in the results with what we expected to get based on our calculations in the design phase of the experiment. If our results scatter more than expected ($\pm P_r$ from the design phase), then a closer look is required at the factors affecting the experiment.

The other primary check in the debugging phase of the experiment is a comparison with basic laws (conservation of energy, etc.) or previously reported, well-accepted results from other experimental programs. Agreement should be expected within the overall uncertainty of the two quantities.

Example. As an example of experiment qualification and of data reporting, consider the following case. The students are asked to determine the free convection heat transfer coefficient from horizontal cylinders of finite length. One of the cylinders has a $\frac{1}{8}$ in. (0.00318 m) diameter and is 6 in. (0.152 m) long for a length to diameter ratio of $L/D = 48$.

The experiment is conducted using the steady state technique. The cylinders are electrically heated and the difference between the cylinder surface temperature and the air temperature is measured directly with a differential thermocouple circuit. For this case of no forced air flow, radiation effects are important and are considered in the data reduction. The precision errors for this experiment were estimated to be negligible with respect to the bias errors, and this was found to be the case when the experiment was performed and several set points were replicated.

The experimental results for the cylinder with $L/D = 48$ are presented in Table 2. While it is important to present the test results in tabular form, a graphical representation is much more useful for observing the full implications of the results. The results are shown in Fig. 6 using the appropriate nondimensional variables (Nusselt number, Nu, and Grashof–Prandtl number, GrPr) and the coordinate system (log–log) which rectifies the data. The 95% confidence uncertainty band is shown for each data point.

Also shown in Fig. 6 is a curve representing the classic results for cylinders with large L/D ($L/D \rightarrow \infty$) from Morgan [8]. After reviewing the data used to obtain the classic results, we concluded that assuming an uncertainty of ±15% about the nominal curve seemed appropriate. This band is shown around the correlation curve in Fig. 6.

Table 2. Experimental results for free convection heat transfer from a horizontal cylinder with $L/D = 48$ (uncertainty values are for $U_h \approx B_h$ and are at a 95% confidence level)

Surface-to-air temperature difference ΔT (K)	Convective heat transfer coefficient h ($W/m^2 K$)	Uncertainty U_h ($W/m^2 K$)
4.4	12.1	1.6
4.5	11.7	1.5
6.3	12.8	1.2
6.3	12.7	1.2
9.2	13.8	0.9
11.8	14.4	0.8
11.8	14.6	0.8
15.1	15.2	0.7
18.0	15.7	0.6
18.2	15.7	0.6
20.9	16.4	0.6
25.6	16.7	0.6

The following points can be made concerning the data presented in Fig. 6. First, the comparison between the experimental data and the classic results for $L/D \rightarrow \infty$ is excellent. The data and uncertainty intervals from the experiment fall within the uncertainty band around the correlation. As anticipated, this long cylinder ($L/D = 48$) approximates an infinite cylinder and provides the opportunity for a check against 'known' values.

Also as shown in Fig. 6, the experiment was run so that four experimental set points were replicated. The small scatter in these results relative to the size of the uncertainty intervals shows that for this experiment the precision errors are much less than the bias errors. This comparison thus confirms the estimates made in the initial phases of the experiment that precision errors were negligible.

CONCLUSION

We have attempted to give an overview of the applications of uncertainty analysis in undergraduate engineering laboratories. We have used examples to try to illustrate the key points of our presentation.

Our objective in the Mechanical Engineering laboratory sequence at Mississippi State University is to present a logical approach to experimentation through the application of uncertainty analysis in the planning, design, construction, debugging, execution, data analysis, and reporting phases of experimental programs. We carry the students through these steps in the second course in the laboratory sequence by planning, designing, and running four experiments and analyzing and reporting the results. The students then apply these principles in the final measurements laboratory.

The book [2] used in this laboratory was developed from notes used and revised by the authors over an 8-year period. The book has also been used as the basis of a graduate course taken by master's and doctoral students.

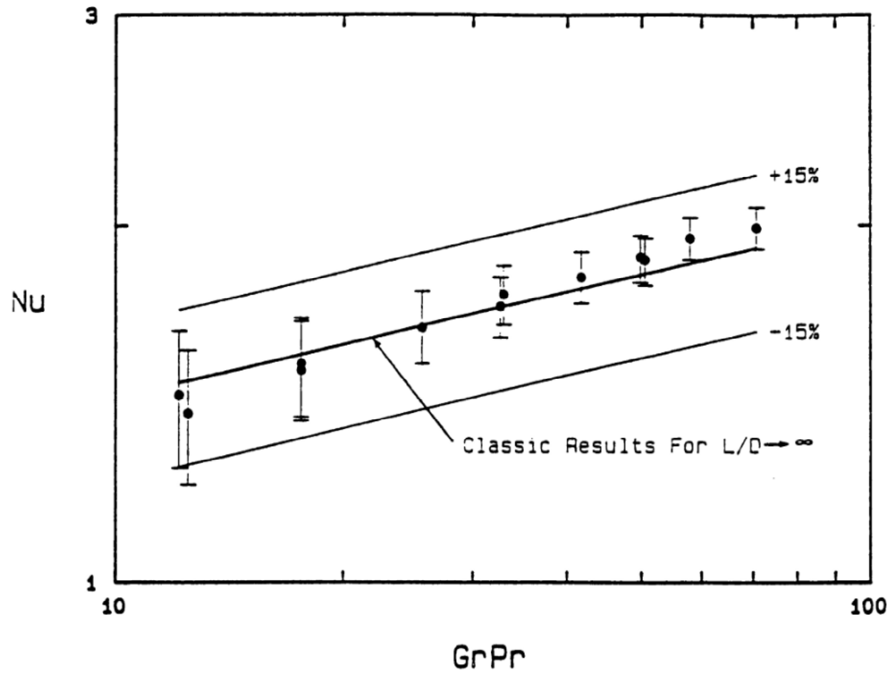

Fig. 6. Free convection heat transfer data for a horizontal cylinder with L/D = 48 (nondimensional rectified coordinates).

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