

# Mathcad in Teaching Rotor and Structural Dynamics\*

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*This paper describes how Mathcad is used to deepen understanding of fundamental aspects in the field of rotor and structural dynamics in two of the undergraduate courses in the education program of Bachelor of Science in Mechanical Engineering with emphasis on Product Development at the University of Karlskrona/Ronneby, Karlskrona, Sweden. Integrating this mathematical software has clearly given an improved interest for mathematics as well as mechanics among the students.*

## INTRODUCTION

THE MACHINE Elements Course is given in the second year of the Bachelor program. Rotor dynamics is a main part of this course. Other parts are springs, connections, bearings, lubrication theories, and brakes. The class-room hours are 28 for lectures and 28 for supervised individual practice on sample problems. When problems of common interest occur the teacher deals with them on the white board in a seminar-like manner. Throughout the course students are also performing a larger compulsory rotor calculation assignment. With the theories supplied at lectures they individually calculate the critical angular speed of a given rotor. Individual guidance for this assignment is given in the exercise hours, but only as answers to questions raised, since one of the general ideas with the assignment is to make each student seek the knowledge necessary to solve the task. The *Mathcad* [2] features of matrix handling, such as eigenvalue calculations, fits perfectly for this assignment, making it possible for the students to advance further in learning dynamics than they otherwise would have managed.

The Structural Dynamics Course is given in the third year of the Bachelor program. The class-room hours are 16 for lectures, 32 for supervised individual practice on sample problems and 8 hours in preparatory work learning the *Mathcad* software, I-DEAS Master Series [3] as the dynamic simulation tool, and the experimental facilities in the structural mechanics lab. After this introduction the students have full access to all the necessary equipment for solving their problems. The same idea of problem-oriented learning as above is applied. Throughout the course problems are given without any answers. Even the textbook lacks answers. Instead the students should present solved problems in a weekly seminar where the other students bring their questions. The teacher

has to explain and help if the solution is not correct. Normally the students come to the teacher and ask for guidance before the seminar, which was the intention.

In addition to the textbook sample problems some other limited problems are given to the students as real physical models, which have to be analyzed in three main steps: theoretical modeling, computer-based simulation and experimental verification. *Mathcad* is here used to solve differential equations, especially to calculate the dynamic response to different excitations with the Laplace transform and to correlate the simulation results and the experimental verification. *Mathcad* is also used as the report writing tool.

## ROTOR DYNAMICS ASSIGNMENT IN THE MACHINE ELEMENTS COURSE

The assignment is presented for the students in the first week of the course and is then supposed to be solved within the seven weeks of course duration. Each student has the total responsibility to hand over a carefully written report including, in all aspects, a correct solution before a stated date. Students who fail to do this are not allowed to participate in the ordinary theoretical test at the end of the course and must perform a new assignment, when the course is repeated next year, in order to achieve their final grades. This rule has turned out to function very well, not the least in making the students active in demanding the teacher to share his knowledge. Only very few students have failed so far. During the work some intermediate results may be checked from the teachers solution. Cooperation between students is allowed and encouraged, but each student has an individual set of data and must produce an individual report. The students have some *Mathcad* experience from prior courses in mathematics. In this course they therefore achieve the necessary additional knowledge by consulting the manual

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and/or the teacher whenever problems occur. This is probably one of the most effective ways of learning this kind of software. A translated version of the 1995 year assignment sheet follows.

*Calculation assignment in the Machine Elements Course 1995*

Figure 1 shows a rotor from a gear box. The shaft has diameter  $d = 30$  mm, Young's modulus  $E = 2.06 \times 10^{11}$  N/m<sup>2</sup> and density  $\rho = 7800$  kg/m<sup>3</sup>. The two cog-wheels have masses  $m_1$  and  $m_2$  and are placed on the shaft according to  $x_1$  and  $x_2$  respectively. The rotor is supported by two bearings at a distance of  $L = 0.5$  m. The bearings together with their surrounding support have linear stiffnesses  $c_1$  and  $c_2$  respectively in all directions perpendicular to the shaft. The left bearing also supports a bending moment proportional to the alignment of the shaft in this bearing, that is  $M = c_M \cdot \theta$ . The right bearing cannot support bending moment.

**Question 1.**

Calculate the ordinary critical angular speeds with respect to bending deformation of the shaft. Consider the cog-wheels as point masses. Distribute the mass of the shaft to the point masses and the bearings. After that the shaft is considered mass-less. The inertia of the mass distributed to the bearings is neglected.

**Question 2.**

For the critical angular speeds (eigenvalues), calculate the deflections of the shaft at the

masses and bearings in relation to the deflection at mass  $m_2$  (eigenvectors). Draw sketches of the shaft appearance.

**Question 3.**

Make a more accurate analysis by representing the shaft as a (higher) number of point masses on an otherwise mass-less shaft. Calculate the four lowest ordinary critical angular speeds with respect to bending deformation of the shaft, with an estimated error below one percent. Use for example Mathcad as a calculation tool.

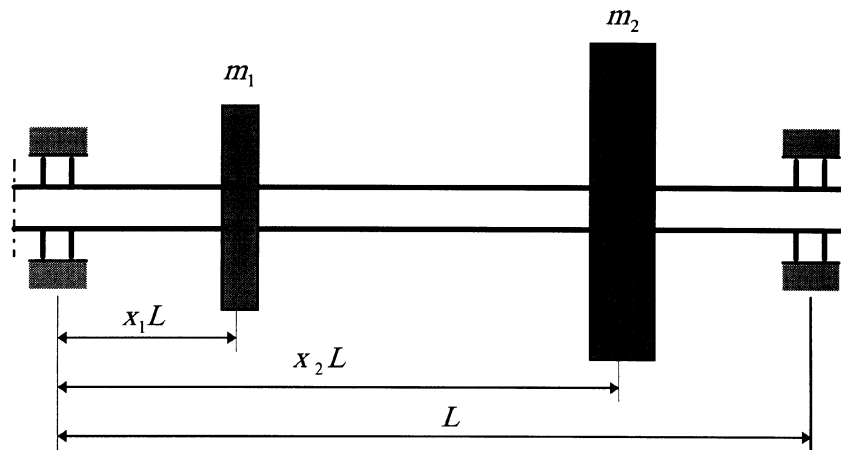
*Comments on the rotor dynamics assignment*

After some work with the laws of dynamics and solid mechanics, Question 1 gives rise to a homogenous system of equations, equation (1):

$$\begin{bmatrix} \alpha_{11}m_1 - \frac{1}{\omega^2} & \alpha_{12}m_2 \\ \alpha_{21}m_1 & \alpha_{22}m_2 - \frac{1}{\omega^2} \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (1)$$

where  $\alpha_{ij}$  = flexibility numbers of the system (inverted stiffness);  $m_i$  = point masses;  $w$  = angular frequency of the shaft;  $y_i$  = deflections at point masses;  $i = 1 \dots 2$ .

The eigenvalues of this matrix is easily determined by pen, paper and pocket-calculator. This method is however recommended to be complemented by Mathcad. Although not necessary it gives a good opportunity to check the results, both ways, and to get familiar with the software for a relatively simple problem. Mathcad may also be



**Data**

$x_1 =$	$m_1 =$	kg	$c_1 =$	N/m
$x_2 =$	$m_2 =$	kg	$c_2 =$	N/m
			$c_M =$	Nm

Fig. 1. Assignment in machine elements course: rotor supported by two bearings.

used for example in calculation of the alpha-values.

Question 2 gives a visualization of the physical meaning of eigenvalues and eigenvectors. To achieve the accuracy required in Question 3 the size of the matrix widely exceeds what is possible to deal with by pen and paper. To solve this question some kind of computer aid is necessary. The Mathcad features of matrix handling fits perfectly for this task.

$$s = \frac{1}{\omega^2} \quad (2)$$

$$A = \begin{bmatrix} \alpha_{11}m_1 & \alpha_{12}m_2 & \cdots & \alpha_{1n}m_n \\ \alpha_{21}m_1 & \alpha_{22}m_2 & \cdots & \alpha_{2n}m_n \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{n1}m_1 & \alpha_{n2}m_2 & \cdots & \alpha_{nn}m_n \end{bmatrix} \quad (3)$$

$$Y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \quad (4)$$

where  $n$  is the number point masses, and with  $I$  as the unity matrix, the eigenvalue problem can be described as:

$$(A - sI)Y = 0 \quad (5)$$

The Mathcad solution then becomes:

$$s := \text{eigenvals}(A) \quad (6)$$

and the critical angular frequencies are obtained from equation (2). The number of point masses is doubled until the difference between two successive iterations is negligible.

### PROBLEMS IN THE STRUCTURAL DYNAMICS COURSE

In parallel with the lectures in the Structural Dynamics Course, the students have to analyze

dynamic characteristics such as natural frequencies and mode shapes of some simple structures considering damping and realistic boundary conditions. The problems are presented as real mechanical models so the answer has to be found with experimental testing. The reports must consist of three parts:

- theoretical modeling
- computer based simulation
- experimental verification.

Two of the problems are described below, followed by a brief example of the solution given by the students.

#### Problem 1.

A circular solid shaft is connected with a solid metal sphere and is connected to a plate according to Fig. 2. Describe the dynamic behavior of the system if the plate is secured to a rigid surface. The material in the model is plain carbon steel.

#### Solution example.

Since the system has one dominant natural frequency the system is modeled as a damped SDOF system with lumped parameters. The approach with the generalized parameter model, Craig [1], can be used:

$$m\ddot{\nu} + c\dot{\nu} + (k - k_g)\nu = 0 \quad (7)$$

where  $\nu$  is a function of location and time  $t$ . The general assumed-modes form in equation (8) is used to solve the differential equation (7):

$$\nu(x, t) = \psi(x)\nu(t) \quad (8)$$

As the assumed-mode shape function the static deflection under a tip force is used:

$$\psi(x) = \frac{x^2}{2L^2} - \frac{x^3}{6L^3} \quad (9)$$

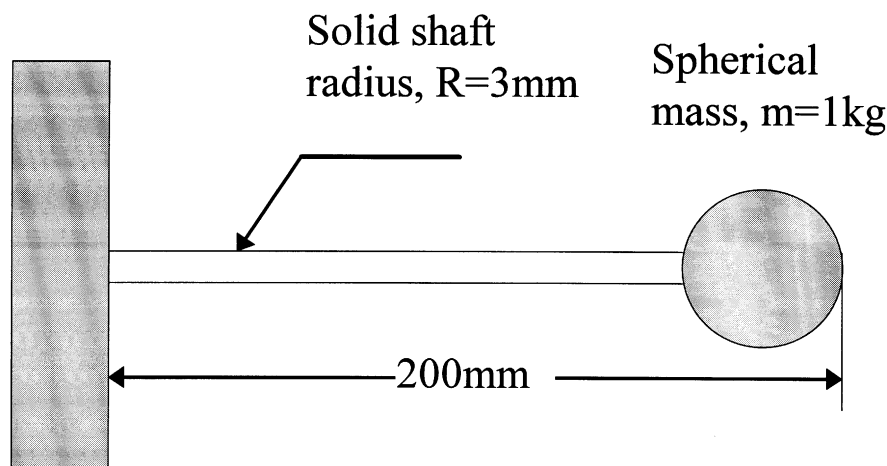


Fig. 2. A solid metal sphere connected to a circular solid shaft.

The generalized parameters can then be calculated as follows:

$$\begin{aligned}
 m &= \int_0^L \rho A \psi^2(x) dx + m_s \psi^2(L) \\
 c &= c_{cr} \zeta \\
 k &= \int_0^L EI (\psi'')^2 dx \\
 k_g &= \int_0^L m_s (\psi')^2 dx
 \end{aligned} \tag{10}$$

where  $c_{cr}$  is the critical damping coefficient. The natural frequency is calculated from:

$$f_n = \frac{1}{2\pi} \sqrt{\frac{(k - k_g)}{m}} \tag{11}$$

The calculation of the integrals and derivatives are performed using Mathcad.

To calculate the response of the system, the damping characteristics have to be determined experimentally. The viscous damping coefficient  $\zeta$  is estimated using the half amplitude method. An impact excitation is performed on the model while recording the response from the laser vibrometer into a PC measuring system, HP 3565. The response curves are imported to Mathcad and analyzed.

To calculate the response from an impact excitation the Laplace transform capabilities in Mathcad are used. Unfortunately Mathcad can't transform a symbolic differential function, but this is the easier part.

$$\begin{aligned}
 m\ddot{v} + c\dot{v} + (k - k_g)v &= p(t) \xrightarrow{\text{Laplace}} ms^2 V + csV \\
 &+ (k - k_g)V = P(s)
 \end{aligned} \tag{12}$$

where  $p(t)$  is the excitation force. The transfer function of the system becomes:

$$H(s) = \frac{1}{ms^2 + cs + (k - k_g)} \tag{13}$$

since the excitation can be approximated with a Dirac pulse this is also the response:

$$V(s) = H(s)P(s) = \frac{1}{ms^2 + cs + (k - k_g)} \tag{14}$$

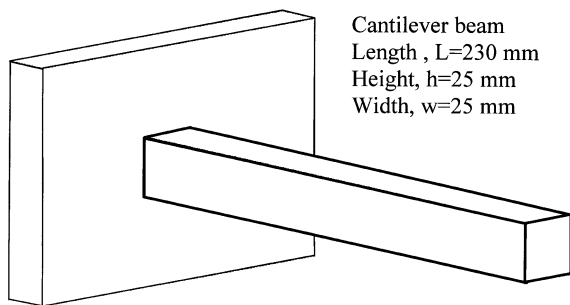


Fig. 3. A solid beam made of plain carbon steel.

The corresponding time response is calculated, using the inverse Laplace transform function, to be:

$$\nu(t) = 2e^{-(c/2m)t} \frac{\sin \left[ \frac{\sqrt{4m(k - k_g) - c^2}}{2m} t \right]}{\sqrt{4m(k - k_g) - c^2}} \tag{15}$$

Diagrams of both the measured and the calculated response are then presented and the correlation between them discussed.

**Problem 2.**

A solid beam made of plain carbon steel is to be analyzed, see Fig. 3. The dimensions and boundary conditions appears from the experimental model according to the figure. Determine the following:

- Natural frequencies and corresponding mode shapes.
- What should the dimension of the cross-section be if the first natural frequency in the width-direction should be half of the natural frequency in the height-direction?

**Solution example.**

The same method as in Problem 1 is often used to calculate the first natural frequency and the deformation shape for this mode. Since this system doesn't have one clearly dominant natural frequency as in Problem 1 the system has to be analyzed as a MDOF system. The system is analyzed as free undamped vibration of an uniform cantilever beam. The equation of motion becomes, Craig [1]:

$$(EI\nu'')'' + \rho A \ddot{v} = 0 \tag{16}$$

Assuming a harmonic motion given by the equation:

$$\nu(x, t) = \psi(x) \cos(\omega t - \alpha) \tag{17}$$

and substitute this into equation (16) gives us the eigenvalue equation:

$$\frac{d^4 \psi}{dx^4} - \lambda^4 \psi = 0 \tag{18}$$

where

$$\lambda^4 = \frac{(\rho A \omega^2)}{EI} \tag{19}$$

the general solution is given by:

$$\begin{aligned}
 \psi(x) &= C_1 \sinh(\lambda x) + C_2 \cosh(\lambda x) \\
 &+ C_3 \sin(\lambda x) + C_4 \cos(\lambda x)
 \end{aligned} \tag{20}$$

The boundary conditions are:

$$\begin{aligned}
 \psi(0) &= 0, & \frac{d\psi}{dx}(0) &= 0, \\
 \frac{d^2 \psi}{dx^2}(L) &= 0, & \frac{d^3 \psi}{dx^3}(L) &= 0
 \end{aligned} \tag{21}$$

Determine the natural frequencies and mode shapes of a uniform cantilever beam

$$E := 205 \cdot 10^9$$

$$L := 0.23$$

$$\rho := 7800$$

$$I := \frac{0.025^4}{12}$$

$$A := 0.025^2$$

Solutions to the characteristic equation (see text)

$$\lambda_1 := \frac{1.8751}{L}$$

$$\lambda_2 := \frac{4.6941}{L}$$

$$\lambda_3 := \frac{7.8548}{L}$$

$$r := 1..3$$

Natural frequencies

$$\omega_r := \frac{(\lambda_r \cdot L)^2}{L^2} \cdot \left( \frac{E \cdot I}{\rho \cdot A} \right)^{\frac{1}{2}}$$

$$f_r := \frac{\omega_r}{2 \cdot \pi}$$

Calculation of results, see text for explanation

$$k_r := \frac{\cosh(\lambda_r \cdot L) + \cos(\lambda_r \cdot L)}{\sinh(\lambda_r \cdot L) + \sin(\lambda_r \cdot L)}$$

$$C := 0.5 \quad \text{Amplification factor}$$

First mode shape

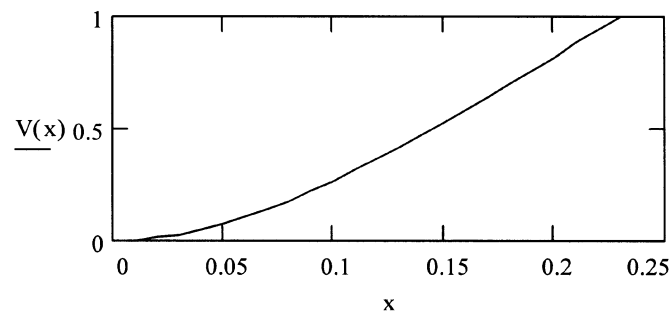
$$r := 1$$

$$x := 0, 0.01 .. L$$

$$V(x) := C \cdot \left[ \cosh(\lambda_r \cdot x) - \cos(\lambda_r \cdot x) - k_r \cdot (\sinh(\lambda_r \cdot x) - \sin(\lambda_r \cdot x)) \right]$$

$$\omega_r = 2.459 \cdot 10^3$$

$$f_r = 391.374$$



(a)

Fig. 4. Example of Mathcad calculation.

**Second mode shape**

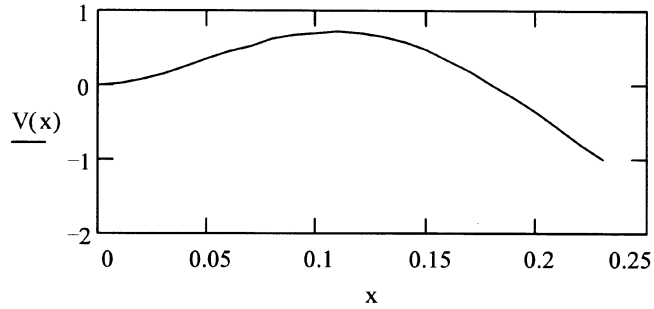
$r := 2$

$x := 0, 0.01 .. L$

$V(x) := C \cdot [\cosh(\lambda_r \cdot x) - \cos(\lambda_r \cdot x) - k_r \cdot (\sinh(\lambda_r \cdot x) - \sin(\lambda_r \cdot x))]$

$\omega_r = 1.541 \cdot 10^4$

$f_r = 2.453 \cdot 10^3$



**Third mode shape**

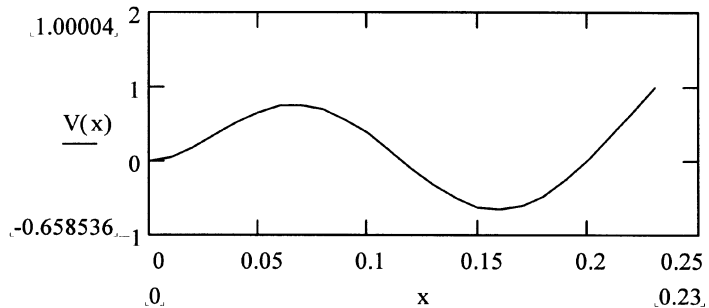
$r := 3$

$x := 0, 0.01 .. L$

$V(x) := C \cdot [\cosh(\lambda_r \cdot x) - \cos(\lambda_r \cdot x) - k_r \cdot (\sinh(\lambda_r \cdot x) - \sin(\lambda_r \cdot x))]$

$\omega_r = 4.315 \cdot 10^4$

$f_r = 6.868 \cdot 10^3$



(b)

Fig. 4. (Continued).

After some work this leads to:

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ \lambda & 0 & \lambda & 0 \\ \lambda^2 \sinh(\lambda L) & \lambda^2 \cosh(\lambda L) & -\lambda^2 \sin(\lambda L) & -\lambda^2 \cos(\lambda L) \\ \lambda^3 \cosh(\lambda L) & \lambda^3 \sinh(\lambda L) & -\lambda^3 \cos(\lambda L) & \lambda^3 \sin(\lambda L) \end{bmatrix} \times \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (22)$$

for the non-trivial solution we found the characteristic equation:

$\cos(\lambda L) \cosh(\lambda L) + 1 = 0 \quad (23)$

whose roots are the eigenvalues  $\lambda_r L$ . Values of  $\lambda_r L$

can be found in the textbook which leads to equation (24) for the natural frequencies:

$\omega_r = \frac{(\lambda_r L)^2}{L^2} \left( \frac{EI}{\rho A} \right)^{1/2} \quad (24)$

The normalized mode shapes can after some work be calculated by the equation:

$\psi_r(x) = \cosh(\lambda_r x) - \cos(\lambda_r x) - k_r [\sinh(\lambda_r x) - \sin(\lambda_r x)] \quad (25)$

where

$k_r = \frac{\cosh(\lambda_r L) + \cos(\lambda_r L)}{\sinh(\lambda_r L) + \sin(\lambda_r L)} \quad (26)$

The resulting mode shapes are then plotted in Mathcad and compared with the simulated results from I-DEAS Simulation. As an example, a sample calculation is presented in Fig. 4 showing how the assignment is performed in Mathcad.

### CONCLUSIONS

After introducing the mathematical software Mathcad in our courses we notice an improved interest for mathematics as well as mechanics among the students. The fear of approaching problems leading to advanced mathematics has clearly decreased. This type of mathematical software gives the opportunity to go further in the study of dynamics than otherwise possible. The

problems described gives the students a feeling of how to correlate simulated and measured results with an analytical model. The awareness of the differences between defining the boundary conditions theoretically and the possibilities to actually fasten structures in practice is very important. Also, presenting problems without written answers makes the students, after some time, confident in their way of working and their own results. They get a good training in critically analyzing their results, which is what reality is about. The poor possibilities of working with, for example, loops and conditions as well as the lack of symbolic Laplace transforms in Mathcad has been experienced as disadvantageous. We are happy to conclude that some of these and other drawbacks are eliminated in the latest update of the software.

### REFERENCES

1. R. R. Jr. Craig, Structural Dynamics, *An Introduction to Computer Methods*, The University of Texas at Austin, J. Wiley & Sons (1981).
2. Mathcad PLUS 5.0, MathSoft, Inc., 101 Main Street, Cambridge, Massachusetts, USA.
3. I-DEAS Master Series 2.1, SDRC, 2000 Eastman Drive, Milford, Ohio 45150, USA.

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