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An Alternative Method for Teaching Dynamics*

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This paper describes an alternative method for teaching undergraduate dynamics that has been used in the Faculty of Mechanical Engineering at Technion for over twenty years. The sequence of topics in the course presents the kinematics of particles, systems of particles and rigid bodies in three dimensions before discussing the kinetics of these systems. This alternative sequence provides the students with sufficient time and practice to master the concepts of a rotating coordinate system which are essential for three-dimensional problems in dynamics. In addition, the paper presents a discussion of indicial notation and tensors, a simple proof of the formula relating the time derivative of rotating base vectors and the angular velocity vector, as well as a convenient tabular notation that helps formulate complicated problems in dynamics. Two example problems are presented to demonstrate the use of this tabular notation.

INTRODUCTION

MOST UNDERGRADUATE courses in dynamics use textbooks [1–5] which introduce the notions of dynamics in an order that is based on increasing mathematical complexity. Specifically, the kinematics and kinetics of particles are introduced for motions in one dimension, two dimensions and then in three dimensions. Similarly, the kinematics and kinetics of rigid bodies are introduced for motions in two dimensions and then in three dimensions. This sequence of topics also can readily be adapted to splitting the study of dynamics into two courses: one for particles and rigid bodies in two dimensions, and the second for rigid bodies in three dimensions.

The main objective of this paper is to discuss aspects of an alternative method for teaching dynamics that has been used in the Faculty of Mechanical Engineering at Technion for over twenty years. This alternative method presents the material in a different sequence in order for the student to have sufficient time to develop proficiency in using the vectorial formulation which is essential for three-dimensional problems. The dynamics course at Technion is the most intense mandatory course in the undergraduate mechanical engineering curriculum. It is taught as a 5.0 credit course with two two-hour lectures and one two-hour recitation per week for a total of fourteen weeks. The lectures are given by professors to large groups of students (50–70) and they cover the basic theory and illustrative examples. The recitations are given by graduate student teaching assistants to smaller groups of students (25–35) and they present detailed solutions of additional example problems.

An outline of this paper is as follows. Section 2 presents an alternative sequence of topics used for teaching dynamics. Section 3 presents basic fundamentals of indicial notation, vectors and tensors. Section 4 discusses a simple proof of the formula relating the time derivative of rotating base vectors and the angular velocity vector, section 5 presents a convenient tabular notation for formulating problems in dynamics, and section 6 presents two example problems. Section 7 presents conclusions. Throughout the text, bold faced symbols are used to denote vectors and tensors. The components of these tensors are presented using indicial notation. Also, $\mathbf{a} \cdot \mathbf{b}$ denotes the dot product and $\mathbf{a} \times \mathbf{b}$ denotes the cross product between two vectors **a** and **b**.

SEQUENCE OF TOPICS

The dynamics course has a unique position in the mechanical engineering curriculum. In spite of the fact that it presents difficult challenges to the analytical abilities and the physical insights of students, it also remains extremely interesting and rewarding. In this regard, the dynamics course offers the opportunity to teach important fundamental aspects of the mathematics of tensors in an environment in which students see the physical relevance and are highly motivated. This course is studied by Technion students in their third or fourth semester, and the main prerequisites are two semesters of calculus, one semester of linear algebra and one semester of statics. These mathematics courses cover differentiation and integration in one, two and three dimensions as well as vectors, matrices, eigenvectors

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Table 1. Sequence of topics for the dynamics course at Technion. (Each week corresponds to two 2-hour lectures and one 2-hour recitation)

Week	Topics
1	Introduction; Vector algebra and indicial notation; Vector calculus; Position, velocity, acceleration; Tangential and normal coordinates; Rectilinear motion
2	Polar coordinates; Cylindrical polar coordinates; Relative motion; Rotating coordinate axes and angular velocity
3	General differential operator; Spherical polar coordinates
4	General rigid body motion
5	Instantaneous screw motion of a rigid body; Contact of bodies
6	Kinetics of a particle
7	Vibrations; Mechanical power, work and energy (particle); Conservative force fields
8	Energy equation for a particle; Angular momentum; Conservation of momentum (yes or no?); Impulse and momentum
9	Kinetics of systems of particles; Alternative formulation of the balance laws; Impulse and momentum (systems of particles); Mechanical power and kinetic energy (systems of particles); Coefficient of restitution
10	Equations of motion of a rigid body; Inertia tensor
11	Inertia tensor (continued); Transfer theorem for the inertia tensor
12	Planar motion
13	Impulse on a rigid body; Energy equation for a rigid body; Angular momentum and transformation relations; Point masses, massless links, and a system of rigid bodies
14	Gyroscopic effects; Euler angles and a spinning top; Euler equations of motion

and eigenvalues. The statics course covers basic equilibrium of bodies, trusses, and machine parts in three dimensions, as well as elasticity of bars in uniaxial stress. Also, the dynamics course is taught in the same semester that the students are learning their first course in differential equations. This means that most of the emphasis is focused on the proper formulation of problems in dynamics and on the simple momentum and energy integrals, instead of on the solution of the resulting nonlinear equations. Nevertheless, solutions of important problems are presented and the results are differentiated to show that these solutions satisfy the equations of motion.

Since the motions of particles, systems of particles and rigid bodies in three dimensions are presented in a single intensive course, it is possible to teach the course using a sequence of topics that is not standard. Table 1 presents an outline of the topics that are taught. From this table it can be seen that the course is divided into two main parts: the first five weeks concentrate on the kinematics of particles, systems of particles and rigid bodies in three dimensions, and the remaining nine weeks concentrate on the kinetics of these systems of bodies.

In the more standard sequence of topics, students can rely on simple geometry and trigonometry to solve problems of particles and rigid bodies in one and two dimensions. Although this geometrical approach is helpful to develop intuition and should be emphasized whenever possible, it cannot be easily generalized to three-dimensional problems which require a vectorial approach. Consequently, students who rely too heavily on the geometrical approach are required to learn both the mathematics of the vectorial approach and the kinetics of rigid bodies in three dimensions near the end of the semester when there is pressure from exams and not enough time to absorb and exercise this difficult material.

In contrast, the sequence of topics at Technion forces the students to become familiar with the vectorial approach near the beginning of the semester by focusing on three-dimensional problems. In particular, the complicated kinematical techniques that are learned during the first part of the semester are exercised continually during the remainder of the semester because the study of kinetics requires the calculation of acceleration.

Moreover, once students have learned the three-dimensional kinematics of a rotating coordinate system to describe the motion of a particle in space, it is trivial to describe the kinematics of a rigid body in three dimensions. Specifically, it is easy for students to understand the fundamental notion of a rigid body which requires the distance between any two material points to remain constant. They also understand that the angle between any two material lines in a rigid body remains constant. Therefore, the base vectors of a body coordinate system which is attached to the rigid body, will remain a set of orthonormal vectors. Consequently, the motion of material points in a rigid body becomes a special case of the motion of a particle relative to a rotating coordinate system.

Before closing this section, it should be mentioned that Suhubi [6] uses a similar sequence of topics in his dynamics course.

INDICIAL NOTATION, VECTORS AND TENSORS

The modern literature in mechanics uses indicial and tensor notation to write mathematical equations for physical laws. Moreover, with the advent of hand-held calculators which easily perform matrix operations, it is very practical and efficient to teach students indicial notation. For these reasons, a few fundamental notions of indicial notation are introduced in the dynamics course. Specifically, the rectangular Cartesian coordinates

(x_1, x_2, x_3) and base vectors $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ are identified with the usual expressions:

$$(x_1, x_2, x_3) = (x, y, z), (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) = (\mathbf{i}, \mathbf{j}, \mathbf{k}) \quad (1a, b)$$

After introducing the notion of a free index (an index that appears only once in an expression and takes the values 1, 2, 3), the components x_i of the position vector $\mathbf{x}$ of a material point are determined by the three indicial equations:

$$x_i = \mathbf{x} \cdot \mathbf{e}_i \quad (2)$$

Next, the notion of a dummy index (an index that appears only twice in an expression) and the Einstein summation convention (a dummy index takes the values 1, 2, 3 and the terms are summed) are introduced so that the vector $\mathbf{x}$ can be expressed in the compact form:

$$\mathbf{x} = x_i \mathbf{e}_i \quad (3)$$

The notion of a second-order tensor appears quite naturally in the discussion of the angular momentum of a rigid body. Specifically, let $\mathbf{x}_B$ (relative to a fixed origin O) locate an arbitrary material point in a rigid body. Then, the position $\mathbf{x}$ (relative to O) of another arbitrary material point in the rigid body can be expressed in the form:

$$\mathbf{x} = \mathbf{x}_B + \boldsymbol{\xi} \quad (4)$$

where the vector $\boldsymbol{\xi}$ locates the point $\mathbf{x}$ relative to B (see Fig. 1). It follows that the angular momentum $\mathbf{H}_B$ of the rigid body relative to the point B can be expressed by the integral:

$$\mathbf{H}_B = \int_P (\mathbf{x} - \mathbf{x}_B) \times \rho(\dot{\mathbf{x}} - \dot{\mathbf{x}}_B)\, dv = \int_P \boldsymbol{\xi} \times \rho\dot{\boldsymbol{\xi}}\, dv \quad (5)$$

where ρ is the mass density (mass per unit volume), P is the region of space occupied by the rigid body, dv is an element of volume of P, and a superposed dot denotes time differentiation.

Before learning kinetics the student has already mastered differentiation with respect to a rotating coordinate system. Consequently, since $\boldsymbol{\xi}$ is a vector connecting two material points on a rigid body, it follows that:

$$\dot{\boldsymbol{\xi}} = \boldsymbol{\omega} \times \boldsymbol{\xi} \quad (6)$$

where $\boldsymbol{\omega}$ is the absolute angular velocity of the rigid body. Moreover, since $\boldsymbol{\omega}$ is a function of time only, it is obvious that $\boldsymbol{\omega}$ can be removed from the integration over space in equation (5).

To this end, the tensor product operator $\otimes$ is defined by the properties of the second order tensor $(\mathbf{a} \otimes \mathbf{b})$:

$$(\mathbf{a} \otimes \mathbf{b})\mathbf{c} = \mathbf{a}(\mathbf{b} \cdot \mathbf{c}), \mathbf{c}(\mathbf{a} \otimes \mathbf{b}) = (\mathbf{c} \cdot \mathbf{a})\mathbf{b},$$
$$(\mathbf{a} \otimes \mathbf{b})^T = \mathbf{b} \otimes \mathbf{a} \quad (7a, b, c)$$

where $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are arbitrary vectors, and $(\mathbf{a} \otimes \mathbf{b})^T$ denotes the transpose of $(\mathbf{a} \otimes \mathbf{b})$. It then follows by using equation (6) in (5), expanding the vector triple product:

$$\boldsymbol{\xi} \times (\boldsymbol{\omega} \times \boldsymbol{\xi}) = (\boldsymbol{\xi} \cdot \boldsymbol{\xi})\boldsymbol{\omega} - (\boldsymbol{\xi} \cdot \boldsymbol{\omega})\boldsymbol{\xi} \quad (8)$$

and using equation (7), that the angular momentum $\mathbf{H}_B$ can be written in terms of the inertia tensor $\mathbf{I}_B$ by the expressions:

$$\mathbf{H}_B = \mathbf{I}_B\boldsymbol{\omega}, \mathbf{I}_B = \int_P \rho[(\boldsymbol{\xi} \cdot \boldsymbol{\xi})\mathbf{I} - \boldsymbol{\xi} \otimes \boldsymbol{\xi}]\, dv = \mathbf{I}_B^T \quad (9a, b)$$

where $\mathbf{I}$ is the second-order unit tensor. Moreover, with the help of the symmetry of $\mathbf{I}$ and the properties of the tensor product it can be seen that $\mathbf{I}_B$ is a symmetric tensor.

Knowledge of the tensor product has the additional important advantage that students can easily recognize how to generalize the representations in equations (2, 3) for vectors to higher-order tensors. In particular, a general second-order tensor $\mathbf{T}$ can be expressed in terms of its components T_{ij} relative to the basis $\mathbf{e}_i$ by recognizing that the nine second-order tensors $(\mathbf{e}_i \otimes \mathbf{e}_j)$ represent the basis for the space of all second-order tensors so that:

$$\mathbf{T} = T_{ij}(\mathbf{e}_i \otimes \mathbf{e}_j), T_{ij} = \mathbf{T} \cdot (\mathbf{e}_i \otimes \mathbf{e}_j) \quad (10a, b)$$

In these expressions the summation convention is employed and the dot product operation has been generalized to second-order tensors so that:

$$(\mathbf{a} \otimes \mathbf{b}) \cdot (\mathbf{c} \otimes \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) \quad (11)$$

where $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ are arbitrary vectors. By comparing the expressions in equations (2, 3, 10) it can be seen that vectors are first-order tensors, which are a special case of general-order tensors.

The knowledge that tensors can be expressed in coordinate-free notation, equations (3, 10a), helps students understand the fundamental nature of transformation relations. In particular, consider

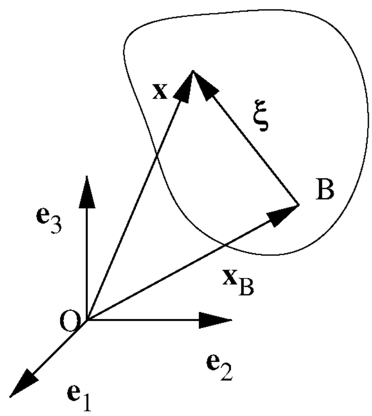

Fig. 1. Kinematics of a rigid body.

another set of right-handed orthonormal base vectors $\mathbf{e}'_i$ which are characterized by their direction cosines A_{ij} relative to the basis $\mathbf{e}_i$ such that:

$$A_{ij} = \mathbf{e}'_i \cdot \mathbf{e}_j = \cos(\mathbf{e}'_i, \mathbf{e}_j) \quad (12)$$

Now, the vector $\mathbf{v}$ and the second-order tensor $\mathbf{T}$ can be expressed in terms of their components v_i and T_{ij} relative to the basis $\mathbf{e}_i$ or in terms of their components v'_i and T'_{ij} relative to the basis $\mathbf{e}'_i$ such that:

$$\mathbf{v} = v_i \mathbf{e}_i = v'_i \mathbf{e}'_i, \; \mathbf{T} = T_{ij}(\mathbf{e}_i \otimes \mathbf{e}_j) = T'_{ij}(\mathbf{e}'_i \otimes \mathbf{e}'_j) \quad (13a, b)$$

These equations express the fact that the tensors $\mathbf{v}$ and $\mathbf{T}$ are coordinate-independent. In contrast, the components of these tensors are determined by the expressions:

$$v_i = \mathbf{v} \cdot \mathbf{e}_i, \; v'_i = \mathbf{v} \cdot \mathbf{e}'_i, \quad (14a, b)$$

$$T_{ij} = \mathbf{T} \cdot (\mathbf{e}_i \otimes \mathbf{e}_j), \; T'_{ij} = \mathbf{T} \cdot (\mathbf{e}'_i \otimes \mathbf{e}'_j) \quad (14c, d)$$

which are seen to be explicitly dependent on the choice of the coordinates. Furthermore, in order to preserve the coordinate-independent nature of the tensors $\mathbf{v}$ and $\mathbf{T}$, the components of these tensors relative to the unprimed basis must be related to those relative to the primed basis by tensor transformation relations.

Using the definition in equation (12) of the direction cosines A_{ij}, these tensor transformation relations can be developed easily by noting that since $\mathbf{e}'_i$ and $\mathbf{e}_i$ both span the three-dimensional space, they can be expressed in the forms:

$$\mathbf{e}'_i = A_{ij} \mathbf{e}_j, \; \mathbf{e}_i = A_{ji} \mathbf{e}'_j \quad (15a, b)$$

Consequently, with the help of equation (14) it follows that:

$$v_i = \mathbf{v} \cdot (A_{ji} \mathbf{e}'_j) = A_{ji} v'_j, \; v'_i = \mathbf{v} \cdot (A_{ij} \mathbf{e}_j) = A_{ij} v_j \quad (16a, b)$$

$$T_{ij} = \mathbf{T} \cdot (A_{mi} \mathbf{e}'_m \otimes A_{nj} \mathbf{e}'_n) = A_{mi} A_{nj} T'_{mn}, \quad (16c)$$

$$T'_{ij} = \mathbf{T} \cdot (A_{im} \mathbf{e}_m \otimes A_{jn} \mathbf{e}_n) = A_{im} A_{jn} T_{mn} \quad (16d)$$

In order to help the students recognize the structure of these results it is important to note that according to the definition in equation (12) of the direction cosines A_{ij}, the first index of A_{ij} is always connected with the primed components, and the second index of A_{ij} is always connected to the unprimed components. Moreover, it is important to emphasize that the order of the indices in the definition (12) represents an arbitrary choice and could have been reversed. However, once the definition for A_{ij} has been made it must be used consistently.

Next, the tensor transformation relations can be rewritten in the more convenient matrix notation by recognizing that the transpose A^T_{ij} of A_{ij} is given by:

$$A^T_{ij} = A_{ji} \quad (17)$$

Consequently, the expressions (16a, c, d) can be rewritten in the forms:

$$v_i = A^T_{ij} v'_j \quad (18a)$$

$$T_{ij} = A^T_{im} T'_{mn} A_{nj}, \; T'_{ij} = A_{im} T_{mn} A^T_{nj} \quad (18b, c)$$

which correspond directly to matrix multiplication. For example, equation (18b), states that the components T_{ij} of $\mathbf{T}$ can be obtained by multiplying the matrix of components T'_{ij} on the left by the transpose of the matrix of A_{ij} and on the right by the matrix of A_{ij}.

Returning to the inertia tensor $\mathbf{I}_B$, it follows that its components I_{Bij} relative to $\mathbf{e}_i$, and its components I'_{Bij} relative to $\mathbf{e}'_i$ become:

$$I_{Bij} = \mathbf{I}_B \cdot (\mathbf{e}_i \otimes \mathbf{e}_j), \; I'_{Bij} = \mathbf{I}_B \cdot (\mathbf{e}'_i \otimes \mathbf{e}'_j) \quad (19a, b)$$

Moreover, since the rigid body moves through space, the region P changes with time so that components I_{Bij} become functions of time. In contrast, if $\mathbf{e}'_i$ are base vectors of a body coordinate system attached to the rigid body, then the components I'_{Bij} of the inertia tensor are constants.

Students of dynamics have difficulty realizing that for many problems it is convenient to choose different coordinate systems and base vectors for simplifying different parts of the problem. However, it is easy for students to understand that the laws of physics cannot depend on any arbitrary mathematical choices, like the choice of coordinates. Consequently, by exposing students to these basic properties of tensors it is possible to emphasize that tensors are the proper mathematical tools to express the laws of physics because they automatically remain coordinate-independent.

THE ANGULAR VELOCITY OF A ROTATING COORDINATE SYSTEM

The base vectors $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z)$ of a cylindrical polar coordinate system are defined relative to the fixed base vectors $\mathbf{e}_i$ in terms of the angle θ such that:

$$\mathbf{e}_r(\theta) = \cos\theta \mathbf{e}_1 + \sin\theta \mathbf{e}_2, \quad \mathbf{e}_\theta(\theta) = -\sin\theta \mathbf{e}_1 + \cos\theta \mathbf{e}_2, \; \mathbf{e}_z = \mathbf{e}_3 \quad (20a, b)$$

It then follows that for a particular particle, θ is a specified function of time and the vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$ rotate counterclockwise about the $\mathbf{e}_z$ axis with angular velocity $\dot{\theta}$. Thus, it is reasonable to define the absolute angular velocity vector $\boldsymbol{\omega}$:

$$\boldsymbol{\omega} = \dot{\theta} \mathbf{e}_z \quad (21)$$

which is characterized by the angular velocity and direction of rotation of the coordinate system. Next, using (20) it can be shown that ω is related to time derivatives of the base vectors by the formulas:

$$\dot{\mathbf{e}}_r = \dot{\theta}\mathbf{e}_\theta = \boldsymbol{\omega} \times \mathbf{e}_r, \quad \dot{\mathbf{e}}_\theta = -\dot{\theta}\mathbf{e}_r = \boldsymbol{\omega} \times \mathbf{e}_\theta, \quad \dot{\mathbf{e}}_z = 0 = \boldsymbol{\omega} \times \mathbf{e}_z \tag{22a, b, c}$$

These results are then used to motivate the case of a general set of rotating base vectors $\mathbf{e}'_i$ which are characterized by the rate equations:

$$\dot{\mathbf{e}}'_i = \boldsymbol{\omega} \times \mathbf{e}'_i \tag{23}$$

where $\boldsymbol{\omega}$ is the absolute angular velocity vector. In most undergraduate books, equation (23) is presented without proof even though $\boldsymbol{\omega}$ no longer remains in a constant direction.

The experience at Technion indicates that students can easily understand the following proof of (23). First, it is noted that since $\mathbf{e}'_i$ are three vectors, they contain a total of nine components. However, since $\mathbf{e}'_i$ are orthonormal vectors, these nine components must satisfy the following six independent constraints:

$$\mathbf{e}'_i \cdot \mathbf{e}'_j = \delta_{ij}, \ \delta_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{24a, b}$$

where δ_{ij} is the Kronecker delta symbol. This means that the base vectors $\mathbf{e}'_i$ contain only three independent components which characterize the three degrees of freedom of rotation. Moreover, using differentiation, these constraints can be written in the rate forms:

$$\dot{\mathbf{e}}'_i \cdot \mathbf{e}'_j + \mathbf{e}'_i \cdot \dot{\mathbf{e}}'_j = 0 \tag{25}$$

Next, equation (23) is substituted into (25) and the properties of the scalar triple product and the cross product are used to write:

$$\boldsymbol{\omega} \times \mathbf{e}'_i \cdot \mathbf{e}'_j + \mathbf{e}'_i \cdot \boldsymbol{\omega} \times \mathbf{e}'_j = \boldsymbol{\omega} \cdot [\mathbf{e}'_i \times \mathbf{e}'_j + \mathbf{e}'_j \times \mathbf{e}'_i] = 0 \tag{26}$$

Since equation (26) is valid for any values of the three components of $\boldsymbol{\omega}$, it follows that the rate equation (23) satisfy the constraints in equation (25), which completes the proof.

Furthermore, for the special case of the base vectors of a cylindrical polar coordinate system which rotate about a fixed axis $\mathbf{e}_z$, it can be shown that the angular velocity vector $\boldsymbol{\omega}$ is also determined by the equation:

$$\boldsymbol{\omega} = \mathbf{e}_r \times \dot{\mathbf{e}}_r, \ \boldsymbol{\omega} = \mathbf{e}_\theta \times \dot{\mathbf{e}}_\theta, \ \text{(no sum on r or } \theta) \tag{27a, b}$$

This suggests that $\boldsymbol{\omega}$ is perpendicular to the plane formed by the base vector and its derivative. However, for the general case where $\boldsymbol{\omega}$ is not in a fixed direction, the equation for $\boldsymbol{\omega}$ must depend on the rotation of all three base vectors in a symmetric manner. Therefore, equations (27) suggest the generalized form:

$$\boldsymbol{\omega} = \tfrac{1}{2}[\mathbf{e}'_i \times \dot{\mathbf{e}}'_i] \tag{28}$$

To prove that equation (28) is valid for the general case, the expression (23) is used together with the properties of the vector triple product to obtain:

$$\begin{aligned} \tfrac{1}{2}[\mathbf{e}'_i \times \dot{\mathbf{e}}'_i] &= \tfrac{1}{2}[\mathbf{e}'_i \times (\boldsymbol{\omega} \times \mathbf{e}'_i)] \\ &= \tfrac{1}{2}[(\mathbf{e}'_i \cdot \mathbf{e}'_i)\boldsymbol{\omega} - (\boldsymbol{\omega} \cdot \mathbf{e}'_i)\mathbf{e}'_i] \\ &= \tfrac{1}{2}[3\boldsymbol{\omega} - \boldsymbol{\omega}] = \boldsymbol{\omega} \end{aligned} \tag{29}$$

Alternatively, equation (23) can be derived by introducing an orthogonal tensor $\mathbf{A}$ with the components A_{ij} (relative to the base vectors $\mathbf{e}_i$) given by equation (12) such that:

$$\mathbf{A} = \mathbf{e}_i \otimes \mathbf{e}'_i, \ \mathbf{A}^T\mathbf{A} = \mathbf{I}, \ \mathbf{A}\mathbf{A}^T = \mathbf{I}, \ \mathbf{e}'_i = \mathbf{A}\mathbf{e}_i \tag{30a, b, c, d}$$

Now, by differentiating the orthogonality condition (30b) it can be shown that the time derivative of $\mathbf{A}$ is related to a skew-symmetric rate of rotation tensor Ω such that:

$$\dot{\mathbf{A}} = \Omega\mathbf{A}, \ \Omega = \dot{\mathbf{A}}\mathbf{A}^T = -\Omega^T \tag{31a, b}$$

Then, differentiation of equation (30d) yields the desired result:

$$\dot{\mathbf{e}}'_i = \Omega\mathbf{A}\mathbf{e}_i = \Omega\mathbf{e}'_i = \boldsymbol{\omega} \times \mathbf{e}'_i \tag{32}$$

where the angular velocity vector $\boldsymbol{\omega}$ is the axial vector of Ω. Although this latter proof is straightforward, it relies on mathematical concepts that are not easily understood by most students in their first course in dynamics. In contrast, the discussion related to equations (24–26) proves the validity of the result (23) and requires less mathematical sophistication, but it does not derive the expression (23) in a straightforward manner. An alternative simple systematic development of the expressions for the angular velocity vector and the time derivatives of rotating base vectors does not yet appear to be available.

A CONVENIENT TABULAR NOTATION

Often in describing complicated three-dimensional motion it is necessary to use one or more sets of rotating base vectors. In particular, let $\mathbf{p}$ be an arbitrary vector that is referred to the base vectors $\mathbf{e}'_i$ which rotate with absolute angular velocity $\boldsymbol{\omega}$ [see equation (23)]:

$$\mathbf{p} = p'_i\mathbf{e}'_i \tag{33}$$

For example, $\mathbf{p}$ could represent the position,

Table 2. Tabular notation for formulating dynamics problems

	1	2	3	4
1		$\mathbf{e}'_1$	$\mathbf{e}'_2$	$\mathbf{e}'_3$
2	ω	ω'_1	ω'_2	ω'_3
3	$\mathbf{x}$	x'_1	x'_2	x'_3
4	$\delta\mathbf{x}/\delta t$	$\dot{x}'_1$	$\dot{x}'_2$	$\dot{x}'_3$
5	$\omega \times \mathbf{x}$	$\omega'_2 x'_3 - \omega'_3 x'_2$	$-\omega'_1 x'_3 + \omega'_3 x'_1$	$\omega'_1 x'_2 - \omega'_2 x'_1$
6	$\mathbf{v} = \dot{\mathbf{x}}$	$v'_1 = \dot{x}'_1 + \omega'_2 x'_3 - \omega'_3 x'_2$	$v'_2 = \dot{x}'_2 - \omega'_1 x'_3 + \omega'_3 x'_1$	$v'_3 = \dot{x}'_3 + \omega'_1 x'_2 - \omega'_2 x'_1$
7	$\delta\mathbf{v}/\delta t$	$\dot{v}'_1$	$\dot{v}'_2$	$\dot{v}'_3$
8	$\omega \times \mathbf{v}$	$\omega'_2 v'_3 - \omega'_3 v'_2$	$-\omega'_1 v'_3 + \omega'_3 v'_1$	$\omega'_1 v'_2 - \omega'_2 v'_1$
9	$\mathbf{a} = \dot{\mathbf{v}}$	$a'_1 = \dot{v}'_1 + \omega'_2 v'_3 - \omega'_3 v'_2$	$a'_2 = \dot{v}'_2 - \omega'_1 v'_3 + \omega'_3 v'_1$	$a'_3 = \dot{v}'_3 + \omega'_1 v'_2 - \omega'_2 v'_1$

velocity, acceleration or the angular momentum of a rigid body. To calculate the time derivative of $\mathbf{p}$ it is necessary to recognize that both the components p'_i and the base vectors $\mathbf{e}'_i$ are functions of time so that:

$$\dot{\mathbf{p}} = \dot{p}'_i \mathbf{e}'_i + p'_i \dot{\mathbf{e}}'_i \tag{34}$$

However, using equations (23) and the notation in [7], this derivative can be written in the form:

$$\dot{\mathbf{p}} = \frac{\delta \mathbf{p}}{\delta t} + \boldsymbol{\omega} \times \mathbf{p} \tag{35}$$

where $\delta\mathbf{p}/\delta t$ is called the frame derivative because it is the time derivative of $\mathbf{p}$ holding the base vectors $\mathbf{e}'_i$ fixed:

$$\frac{\delta \mathbf{p}}{\delta t} = \dot{p}'_i \mathbf{e}'_i \tag{36}$$

The formula in equation (35) is sometimes called the general differential operator.

The late Professor M. Reiner at Technion developed a tabular notation that helps organize the formulation of complicated dynamics problems onto a single page and Dr. B. Popper [8] introduced this notation into the dynamics curriculum. This tabular notation is based on the representation of the cross product as the determinant of a matrix:

$$\boldsymbol{\omega} \times \mathbf{p} = \begin{vmatrix} \mathbf{e}'_1 & \mathbf{e}'_2 & \mathbf{e}'_3 \\ \omega'_1 & \omega'_2 & \omega'_3 \\ p'_1 & p'_2 & p'_3 \end{vmatrix} = (\omega'_2 p'_3 - \omega'_3 p'_2)\mathbf{e}'_1 - (\omega'_1 p'_3 - \omega'_3 p'_1)\mathbf{e}'_2 + (\omega'_1 p'_2 - \omega'_2 p'_1)\mathbf{e}'_3 \tag{37}$$

Table 2 shows an example of this tabular notation where the position vector $\mathbf{x}$, the velocity $\mathbf{v}$, and the acceleration $\mathbf{a}$ have been calculated using their representations relative to the rotating base vectors $\mathbf{e}'_i$. For convenience, the rows and columns have been numbered. Row 1 lists the rotating base vectors, and row 2 lists the components of the absolute angular velocity of these rotating base vectors. Row 3 lists the components of the position vector, and row 4 lists the frame derivative of $\mathbf{x}$. Row 5 lists the result of the determinant of the matrix formed by rows 1, 2 and 3 [see eqn. (37)], and the row 6 lists the velocity $\mathbf{v}$ which is determined by the sum of rows 4 and 5. Similarly, row 7 lists the frame derivative of the velocity $\mathbf{v}$ and row

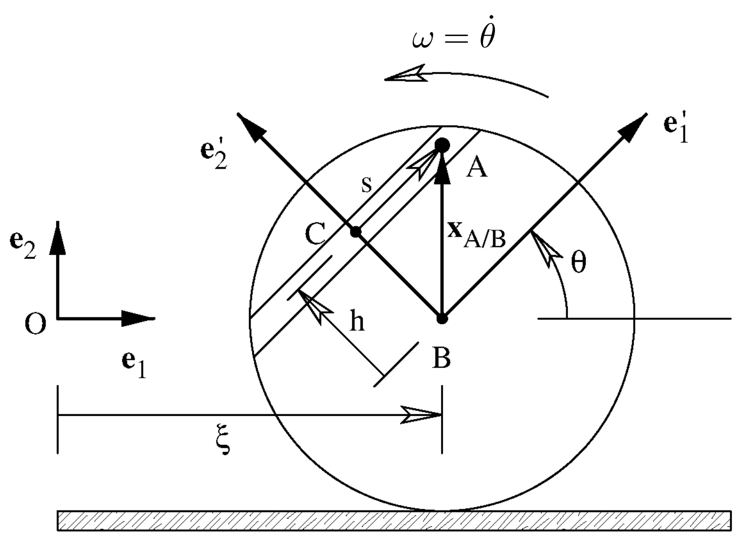

Fig. 2. Sketch of a rolling disk with a moving particle A in the slot CA.

Table 3. Tabular notation associated with the problem in Fig. 2

	$\mathbf{e}_1'$	$\mathbf{e}_2'$	$\mathbf{e}_3'$
ω	0	0	$\omega = \dot{\theta}$
$\mathbf{x}_{A/B}$	s	h	0
$\delta \mathbf{x}_{A/B}/\delta t$	$\dot{s}$	0	0
$\omega \times \mathbf{x}_{A/B}$	$-\omega h$	ωs	0
$\mathbf{v}_{A/B}$	$\dot{s} - \omega h$	ωs	0
$\delta \mathbf{v}_{A/B}/\delta t$	$\ddot{s} - \dot{\omega} h$	$\dot{\omega} s + \omega \dot{s}$	0
$\omega \times \mathbf{v}_{A/B}$	$-\omega(\omega s)$	$\omega(\dot{s} - \omega h)$	0
$\mathbf{a}_{A/B}$	$\ddot{s} - \dot{\omega} h - \omega^2 s$	$\dot{\omega} s + 2\omega \dot{s} - \omega^2 h$	0

8 lists the result of the determinant of the matrix formed by rows 1, 2 and 6 [see eqn. (37)]. Finally, row 9 lists the acceleration **a** which is determined by the sum of rows 7 and 8. It is important to emphasize that the components of all vectors are referred to the base vectors in row 1 of the table. Moreover, individual tables must be used for each set of rotating base vectors.

The experience at Technion indicates that this tabular notation helps the students formulate complicated problems in dynamics because it is very systematic and easy to check.

EXAMPLES

In order to introduce the notion of relative motion it is convenient to consider the problem sketched in Fig. 2. Specifically consider a circular disk which rolls without slipping on a horizontal surface. Particle A is constrained to move in the slot CA which is located a constant distance h from the center B of the disk. The motion of particle A is determined by the function s(t). The base vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ are fixed in space and are chosen so that $\mathbf{e}_1$ is parallel to the horizontal plane. If the angular velocity $\boldsymbol{\omega}$ of the disk vanishes, then it is clear that the motion of A can easily be characterized by the base vectors $\mathbf{e}_1'$ and $\mathbf{e}_2'$ when $\mathbf{e}_1'$ is chosen to be parallel to the slot CA and when the origin is located at the point C. However, if the disk rolls and $\boldsymbol{\omega}$ is nonzero, then the point C exhibits a complicated motion.

Under these conditions, it is still convenient to choose $\mathbf{e}_1'$ to be parallel to the slot but it is more convenient to refer the motion of particle A to the center of the disk B. Specifically, the vector $\mathbf{x}_A$ which locates the point A relative to the fixed origin O, can be expressed in terms of the vector $\mathbf{x}_B$ which locates the point B relative to the fixed origin O, and the vector $\mathbf{x}_{A/B}$ which locates the point A relative to B, such that:

$$\mathbf{x}_A = \mathbf{x}_B + \mathbf{x}_{A/B},\ \mathbf{x}_B = \xi(t)\,\mathbf{e}_1,\quad \mathbf{x}_{A/B} = s(t)\,\mathbf{e}_1' + h\,\mathbf{e}_2' \qquad (38a, b, c)$$

where $\xi(t)$ is the distance between O and B and, because of the no slip condition, its derivative is related to the angular velocity and the radius of the

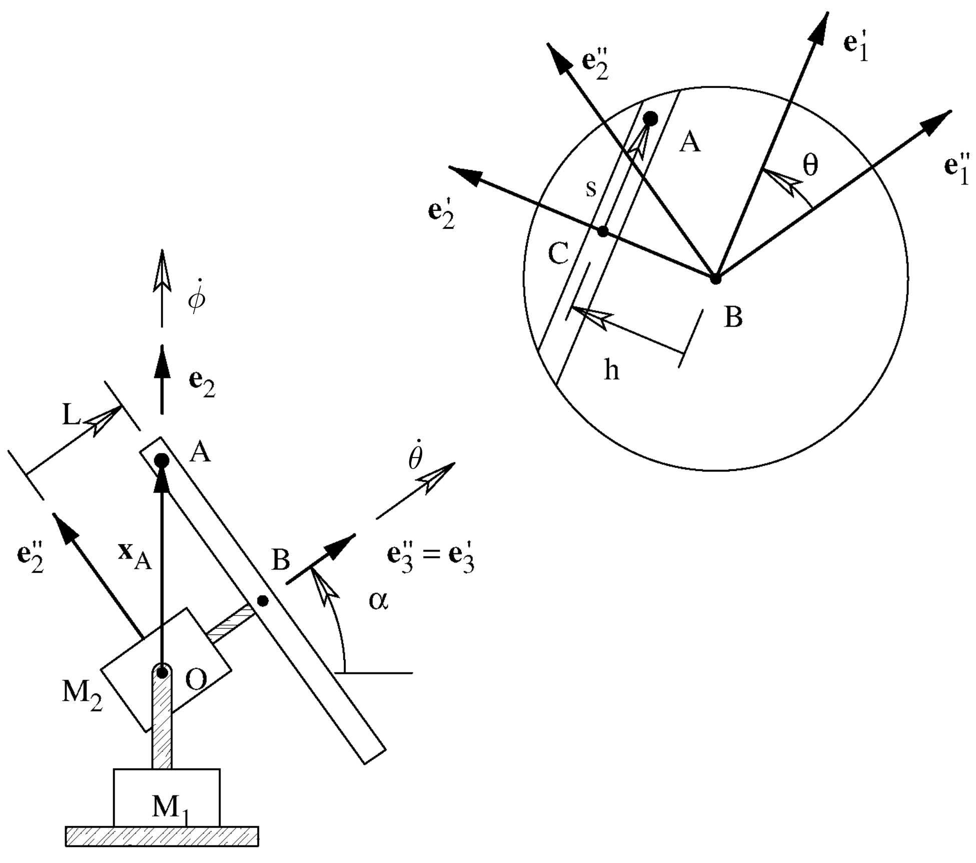

Fig. 3. Sketches of two views of a disk being rotated by two motors.

disk. The absolute velocities $\mathbf{v}_A$ and $\mathbf{v}_B$, and the relative velocity $\mathbf{v}_{A/B}$ are given by:

$$\mathbf{v}_A = \dot{\mathbf{x}}_A,\ \mathbf{v}_B = \dot{\mathbf{x}}_B,\ \mathbf{v}_{A/B} = \dot{\mathbf{x}}_{A/B} \quad (39a, b, c)$$

Similarly, the absolute accelerations $\mathbf{a}_A$ and $\mathbf{a}_B$, and the relative acceleration $\mathbf{a}_{A/B}$ are given by:

$$\mathbf{a}_A = \dot{\mathbf{v}}_A,\ \mathbf{a}_B = \dot{\mathbf{v}}_B,\ \mathbf{a}_{A/B} = \dot{\mathbf{v}}_{A/B} \quad (40a, b, c)$$

This example helps emphasize the point that although the choice of the coordinate system used to formulate the problem is arbitrary, that choice can simplify or complicate the solution. Specifically, for this example it is easiest to differentiate the vectors $\mathbf{x}_B$ and $\mathbf{v}_B$ when they are referred to the fixed base vectors, whereas it is easiest to differentiate the vectors $\mathbf{x}_{A/B}$ and $\mathbf{v}_{A/B}$ when they are referred to the rotating base vectors. Once the differentiation has been performed the resulting vectors can be expressed in terms of either coordinate system by using the transformation relations:

$$\mathbf{e}'_1 = \cos\theta\mathbf{e}_1 + \sin\theta\mathbf{e}_2,\ \mathbf{e}'_2 = -\sin\theta\mathbf{e}_1 + \cos\theta\mathbf{e}_2 \quad (41a, b)$$

$$\mathbf{e}_1 = \cos\theta\mathbf{e}'_1 - \sin\theta\mathbf{e}'_2,\ \mathbf{e}_2 = \sin\theta\mathbf{e}_1 + \cos\theta\mathbf{e}_2 \quad (41c, d)$$

where θ is the angle between $\mathbf{e}_1$ and $\mathbf{e}'_1$ so that $\omega = \dot{\theta}$. In particular, $\mathbf{v}_B$ and $\mathbf{a}_B$ become:

$$\mathbf{v}_B = \dot{\xi}[\cos\theta\mathbf{e}'_1 - \sin\theta\mathbf{e}'_2],\ \mathbf{a}_B = \ddot{\xi}[\cos\theta\mathbf{e}'_1 - \sin\theta\mathbf{e}'_2] \quad (42a, b)$$

Also, Table 3 shows the calculation of the relative velocity $\mathbf{v}_{A/B}$ and acceleration $\mathbf{a}_{A/B}$ using the absolute angular velocity ω of the base vectors $\mathbf{e}'_i$. In this regard, it should be emphasized that the relevant physics and geometry associated with the motion of the body is translated from the free-body diagram into the mathematical expressions in the first three rows of the table. The mathematical manipulations associated with the remaining rows in the table are purely technical and once they have been mastered by the student, they can be programmed symbolically.

As another example, consider the fully three-dimensional problem sketched in Fig. 3 which shows the same disk which is now rotated by two motors. It is obvious that this latter problem cannot be dealt with geometrically. Specifically, the base of motor M_1 is fixed and its shaft rotates with the angular velocity $\dot{\phi}$ relative to its base about the fixed vertical axis $\mathbf{e}_2$. The base of motor M_2 is attached to the shaft of motor M_1 by a joint that allows rotation about the horizontal axis which passes through the fixed origin O. The shaft OB of the motor M_2 has constant length L, and it rotates with angular velocity $\dot{\theta}$ relative to the motor's base. The angle between the shaft of motor M_2 and the horizontal is characterized by $\alpha(t)$.

Table 4. First three rows of the table for formulating the problem in Fig. 3 in terms of the base vectors $\mathbf{e}''_i$.

	$\mathbf{e}''_1$	$\mathbf{e}''_2$	$\mathbf{e}''_3$
Ω	$-\dot{\alpha}$	$\dot{\phi}\cos\alpha$	$\dot{\phi}\sin\alpha$
$\mathbf{x}_A$	$s\cos\theta - h\sin\theta$	$s\sin\theta + h\cos\theta$	L

Finally, the disk is attached rigidly to the shaft of motor M_2 and the particle A moves in the slot CA.

For this problem it is convenient to introduce the base vectors $\mathbf{e}_i$ which are fixed; the base vectors $\mathbf{e}''_i$ which rotate with the base of motor M_2, with $\mathbf{e}''_3$ coinciding with the shaft OB and $\mathbf{e}''_2$ remaining in the vertical plane; and the base vectors $\mathbf{e}'_i$ which rotate with the disk, with $\mathbf{e}'_3$ coinciding with the shaft OB (see Fig. 3). The absolute angular velocities of these rotating base vectors are defined such that:

$$\dot{\mathbf{e}}''_i = \Omega \times \mathbf{e}''_i,\ \Omega = \dot{\phi}\mathbf{e}_2 - \dot{\alpha}\mathbf{e}''_1 \quad (43a, b)$$

$$\dot{\mathbf{e}}'_i = \omega \times \mathbf{e}'_i,\ \omega = \Omega + \dot{\theta}\mathbf{e}''_3 \quad (43c, d)$$

where the angular velocity vector Ω should not be confused with the tensor temporarily introduced in equation (31). The expression (43b) is used to emphasize that vectors can be added when they are expressed with respect to different base vectors even though components of vectors can only be added when they are referred to the same base vectors. Moreover, it is emphasized that the student should first write the vectors in their simplest forms [e.g. referring the angular velocity $\dot{\phi}$ to the direction $\mathbf{e}_2$ and referring the angular velocity $\dot{\alpha}$ to the direction $\mathbf{e}''_1$]. Any transformations to different coordinate systems can be done afterwards as needed. This avoids complicating a free-body diagram with unnecessary components of vectors.

Because of the complexity of this problem, the choice of the coordinate system used to calculate the absolute acceleration of particle A depends to a large extent on individual preference and is not straightforward. For any choice there is some compromise associated with the simplicity of the representation of either the absolute angular velocity or the position vector. However, usually it is convenient to choose an intermediate coordinate system like that associated with the base vectors $\mathbf{e}''_i$. In order to use the base vectors $\mathbf{e}''_i$ it is necessary to write $\mathbf{e}_2$ in terms of $\mathbf{e}''_i$:

$$\mathbf{e}_2 = \cos\alpha\mathbf{e}''_2 + \sin\alpha\mathbf{e}''_3 \quad (44a, b)$$

Table 5. First three rows of the table for formulating the problem in Fig. 3 in terms of the base vectors $\mathbf{e}'_i$

	$\mathbf{e}'_1$	$\mathbf{e}'_2$	$\mathbf{e}'_3$
ω	$-\dot{\alpha}\cos\theta$ $+\dot{\phi}\cos\alpha\sin\theta$	$\dot{\alpha}\sin\theta$ $+\dot{\phi}\cos\alpha\cos\theta$	$\dot{\theta} + \dot{\phi}\sin\alpha$
$\mathbf{x}_A$	s	h	L

Similarly, in order to use the base vectors $\mathbf{e}'_i$ it is necessary to use the transformation relations:

$$\mathbf{e}''_1 = \cos\theta\mathbf{e}'_1 - \sin\theta\mathbf{e}'_2,$$
$$\mathbf{e}''_2 = \sin\theta\mathbf{e}'_1 + \cos\theta\mathbf{e}'_2, \qquad (45\text{a, b, c})$$
$$\mathbf{e}''_2 = \mathbf{e}'_3.$$

Table 4 shows the first three rows of the table for formulating the problem in Fig. 3 in terms of the base vectors $\mathbf{e}''_i$, and Table 5 shows the first three rows of the table for formulating the problem in Fig. 3 in terms of the base vectors $\mathbf{e}'_i$. In both of these tables the vector $\mathbf{x}_A$ locates the position of particle A relative to the fixed origin O. From these tables it can be seen that the angular velocity Ω in Table 4 is much simpler than the angular velocity $\boldsymbol{\omega}$ in Table 5, and that the position vector $\mathbf{x}_A$ in Table 4 is more complicated than that in Table 5. For either of these tables the remaining rows can be obtained in a straightforward manner. However, since the resulting expressions are so complicated they are not shown explicitly.

CONCLUSIONS

The alternative sequence of topics taught at Technion provides the students with sufficient time and practice to master the concepts of a rotating coordinate system. The students are challenged by this approach and they feel a great sense of accomplishment at the end of the semester because they know that they can formulate even the most difficult three-dimensional problems in dynamics. The success of this approach is determined by the fact that the students pass the final exam which is specifically designed to test if they can calculate acceleration of a particle in a coordinate system rotating in three-dimensions, and the rate of change of angular momentum of a rigid body experiencing full three-dimensional motion.

Since this is a first course in dynamics, it is important to develop physical intuition about dynamics. Therefore, a number of two-dimensional examples are presented and analyzed with both the geometrical approach and the vectorial approach. This combined approach is essential to the development of familiarity with the vectorial approach in a setting in which the students can check the equations relative to the geometrical approach with which they are more comfortable in the beginning.

Moreover, during the course the students are exposed to indicial notation and tensors and the fundamental aspects of the coordinate-free nature of tensors is emphasized. However, a conscious effort is made to present only the bare minimum of details related to these subjects. Our experience indicates that undergraduate students have very little trouble mastering the technical aspects of tensor manipulations like those associated with equations (14) and (16). However, it is not expected that they will fully understand the deep properties of tensors at this early stage in their education. The objective here is to present them with correct mathematical tools that will remain valid if they pursue the subject of tensors later in more depth.

It is our opinion, the proposed alternative sequence of topics is appropriate for any dynamics course which has the objective of teaching aspects of rigid body dynamics in three dimensions. Of course, the number of example problems and the range of topics covered will depend on the number of teaching hours that are dedicated to the specific course. Although the Technion's students have been exposed to calculus in their senior year of high school, they typically do not read sections of the textbook independently at home. In contrast, it is our understanding that a typical student in the United States does independent reading assignments. Therefore, it may be possible to adopt this alternative sequence of topics for such students without having to significantly increase the number of contact hours.

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