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The Use of Generalised Functions in the Discontinuous Beam Bending Differential Equations*

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This paper discusses materials for a course in Strength of Materials and Mechanics of Solids, addressed to students of second-year Mechanical Engineering, Ocean Engineering, Civil Engineering and Aerospace Engineering. In particular, the presence of discontinuities in the beam-bending differential equations is considered. This problem is solved by the use of the generalised functions, among which the best known is the Dirac delta function. In particular Macaulay's approach, which uses these functions when discontinuous mechanical loads are present, is here extended to the cases in which discontinuous external loads are present, giving discontinuities on displacements and rotations. Moreover, the cases in which natural and essential constraints are along the beam axis, giving different kinds of discontinuities, are presented. This extension shows the same easy applicability and the same practical advantages of the Macaulay's approach, always reducing to one the differential equations to be solved in order to find the displacements law. Moreover the formulation is given in a uniform way for any kind of discontinuity appearing in the beam. The mode of presentation of this material is by lecture and is run as a regular course. Hours required to cover the arguments are 3 to 4 with 2 to 3 revision hours. This lecture must be held after that the classical beam bending problem has been treated. The new aspects presented in this paper hopefully help the students to find important connections between some aspects of mathematician analysis and an important problem of applied mechanics.

INTRODUCTION

THE BEAM-BENDING differential equations represent a fundamental topic for any course in Strength of Materials and Mechanics of Solids. When the loads acting on the beam are continuous it is well known that the solution of the fourth-order beam-bending differential equation requires the determination of four integration constants. They can be obtained by the solution of four algebraic equations corresponding to the natural and essential boundary conditions on the beam.

When a discontinuous load is applied on the beam, the usual approach consists in writing the beam-bending differential equation for each part of the beam in which the load is continuous. Consequently, if these parts are n, then the constants of integrations to be determined are $4n$ and, obviously, the boundary conditions to be imposed are $4n$, too. Hence, if many discontinuous loads are applied on the beam, this approach can result in being heavy from a practical point of view.

An alternative approach, which is considered only in some textbooks, for example in [1, 2], is Macaulay's method [3], consisting of the use of the so-called generalised (or improper) functions [4], in order to take into account the discontinuities. One of the most used generalised functions in any field of science is the Dirac delta function [5] and all the other generalised functions used in Macaulay's method are its generalised derivatives or integrals. The rules of generalised derivatives and integrations of these functions are rigorously studied in a relatively new discipline known as theory of distributions [6]. The use of Macaulay's method allows us to treat the discontinuous loads as continuous. This implies that, for any distribution of continuous and discontinuous loads, the integration constants to be determined and the boundary conditions to be imposed are always four, with a sure practical advantage. The Macaulay's method is applied in the above quoted textbooks only when the discontinuities are in the mechanical loads.

Brungraber [7] extended Macaulay's method to the case in which along-axis supports (determining essential boundary conditions) and along-axis natural constraints are present.

In this work the use of the generalised functions is extended to the cases in which the discontinuities are in the displacements and rotations. These cases are not usually considered in the textbooks in which Macaulay's method is treated. Even for these cases the use of the generalised functions allows us to consider only one beam-bending differential equation with the consequent practical

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advantage when the integration constants have to be evaluated.

In the first following section the generalised functions and the rules of the generalised derivative and integration are recalled; in the second one the classical Macaulay's method is considered; while in the other three sections it is extended to the other cases of discontinuities that can be present in the beam problem; in the last section an example is presented in order to show the easy applicability of the approach and the practical advantage of its use.

GENERALISED FUNCTIONS

One of the most used generalised function in many fields of sciences is the *Dirac delta* function or *impulse* function $\delta(x-x_0)$. The simplest way to define the Dirac delta function is by means of the following three properties:

$$\begin{aligned} &\delta(x-x_0)=0 \quad x\neq x_0;\\ &\int_{-\infty}^{+\infty}\delta(x-x_0)\,dx=1;\\ &\int_{-\infty}^{+\infty}\delta(x-x_0)f(x)\,dx=f(x_0) \end{aligned} \tag{1}$$

As pointed out in [4], the impulse $\delta(x-x_0)$ does not represent a function in the classical analytical sense. To stress this concept, Dirac himself coined for it the term *improper function* [5], while Temple in 1953 introduced the term *generalised function.* Consequently the above first integral is not a meaningful quantity until a convention for interpreting it is declared. Usually it is used to mean:

$$\lim_{\tau\to 0}\int_{-\infty}^{+\infty}\Pi\left(\frac{x-x_0}{\tau}\right)dx=1 \tag{2}$$

where the function $\Pi((x-x_0)/\tau)$ is a rectangle function of height τ^{-1} and base τ placed at the abscissa x_0, having obviously unitary area, that is:

$$\Pi\left(\frac{x-x_0}{\tau}\right)=\begin{array}{ll} 0, & x<x_0-\dfrac{\tau}{2}\\ 1, & x_0-\dfrac{\tau}{2}<x<x_0+\dfrac{\tau}{2}\\ 0, & x>x_0+\dfrac{\tau}{2}\end{array} \tag{3}$$

In this way the Dirac delta function can be considered as the limit of the rectangle function. Hence every differential or integral operation can be applied on the Dirac delta function, if it is effectively applied on the rectangle function and then the limit is performed.

Keeping this convention in mind, a fundamental relationship between the Dirac delta function and the unit step function $H(x-x_0)$ placed at the abscissa x_0 can be obtained as follows:

$$\int_{-\infty}^{x}\delta(y-x_0)\,dy=H(x-x_0) \tag{4}$$

or as in the following inverse form:

$$\delta(x-x_0)=H'(x-x_0)\equiv H_{,1}(x-x_0) \tag{5}$$

The derivative in this last relationship has to be intended not in a strict analytical sense but in the above defined sense; hence we refer to it as *generalised (or formal) derivative.* In this sense we can say that the Dirac delta is the derivative of the unit step function, or that the unit step function is the integral of the Dirac delta.

In the same way the formal integral of the unit step function can be performed, giving the unit ramp function $R(x-x_0)$, defined as follows:

$$\begin{aligned} &R(x-x_0)=0, \quad x<x_0\\ &R(x-x_0)=x-x_0, \quad x>x_0 \end{aligned} \tag{6}$$

Hence, the following formal relationship can be written:

$$\begin{aligned} &R'(x-x_0)\equiv R_{,1}(x-x_0)=H(x-x_0);\\ &R(x-x_0)=\int_{-\infty}^{x}H(y-x_0)\,dy \end{aligned} \tag{7}$$

On its turn, it is easy to verify that the ramp function can be considered as the generalised derivative of a particular generalised function defined as follows:

$$\begin{aligned} &P(x-x_0)=0 \quad x<x_0\\ &P(x-x_0)=\tfrac{1}{2}(x-x_0)^2 \quad x\geq x_0 \end{aligned} \tag{8}$$

Consequently we write:

$$\begin{aligned} &P'(x-x_0)\equiv P_{,1}(x-x_0)=R(x-x_0);\\ &P(x-x_0)=\int_{-\infty}^{x}R(y-x_0)\,dy \end{aligned} \tag{9}$$

The $P(\cdot)$ function can be defined as *parabolic ramp.*

Again, if a generalised integral is performed on the parabolic ramp $P(x-x_0)$, it is easy to show that the result is a generalised function $C(x-x_0)$, that can be called *cubic ramp*. It is defined as follows:

$$\begin{aligned} &C(x-x_0)=0 \quad x<x_0\\ &C(x-x_0)=\tfrac{1}{6}(x-x_0)^3 \quad x\geq x_0 \end{aligned} \tag{10}$$

and hence:

$$\begin{aligned} &C'(x-x_0)\equiv C_{,1}(x-x_0)=P(x-x_0);\\ &C(x-x_0)=\int_{-\infty}^{x}P(y-x_0)\,dy \end{aligned} \tag{11}$$

A further generalised integral of the cubic ramp arises the *fourth order ramp* $Q(x-x_0)$ that is defined as follows:

$$\begin{aligned} &Q(x-x_0)=0 \quad x<x_0\\ &Q(x-x_0)=\tfrac{1}{24}(x-x_0)^4 \quad x\geq x_0 \end{aligned} \tag{12}$$

Extending these conclusions to the *n*th order generalised integral of the ramp function $R(x-x_0)$, we can affirm that it is the *n*th *order ramp*, indicated as $R_n(x-x_0)$ and defined as follows:

$$\begin{aligned} R_n(x-x_0) &= 0 \quad x < x_0 \\ R_n(x-x_0) &= \frac{1}{n!}(x-x_0)^n \quad n \geq x_0 \end{aligned} \tag{13}$$

Hence the following relationships hold:

$$R_1(x-x_0) \equiv R(x-x_0);$$
$$R_2(x-x_0) \equiv P(x-x_0);$$
$$R_3(x-x_0) \equiv C(x-x_0);$$
$$R_4(x-x_0) \equiv Q(x-x_0).$$

Consequently, we can set $R_0(x-x_0) \equiv H(x-x_0)$.

Even the generalised derivative of the Dirac delta function can be performed and the result is the so-called *doublet*, that is a generalised function consisting of a couple of second-order Dirac delta functions having opposite sign and both placed at the abscissa x_0 (here with the term *n*th order Dirac delta function we refer to the limit of the *nth* order rectangle function $\Pi_n((x-x_0)/\tau)$. Due to this relationship with the Dirac delta function, the doublet is usually indicated as $\delta'(x-x_0)$. The further generalised derivative of the doublet gives the so-called *double-doublet*, that is a generalised function consisting of four fourth-order Dirac delta functions placed at x_0 and is indicated by $\delta''(x-x_0)$.

It is easy to show that the *n*th formal derivative of the Dirac delta function gives a generalised function consisting of 2^n alternate *n*th order Dirac delta functions placed at x_0. This generalised function can be defined as *double-n function*, is indicated as $\delta^{(n)}(x-x_0)$ and the following properties can be verified for it:

$$\int_{-\infty}^{+\infty} \delta^{(n)}(x-x_0)\,dx = 0;$$
$$\int_{-\infty}^{+\infty} \delta^{(n)}(x-x_0) f(x)\,dx = (-1)^n f^{(n)}(x_0) = (-1)^n \left[\frac{d^n f(x)}{dx^n}\right]_{x=x_0} \tag{14}$$

For uniformity's sake, it is convenient indicating these generalised functions as follows:

$$\begin{aligned} R_{-1}(x-x_0) &\equiv \delta(x-x_0); \\ R_{-n-1}(x-x_0) &\equiv \delta^{(n)}(x-x_0) \end{aligned} \tag{15}$$

In this way it is always possible to write:

$$R_i'(x-x_0) \equiv R_{i,1}(x-x_0) = R_{i-1}(x-x_0), \quad i = -n, \ldots, 1, 0, 1, \ldots, n \tag{16}$$

DISCONTINUITIES IN THE LOADS (MACAULAY'S APPROACH)

It is well known that the differential equation governing the transversal displacements $u(x)$ of an elastic bending-beam subjected to the continuous transversal load $p(x)$ has the following form:

$$u''''(x) \equiv u_{,4}(x) = \frac{p(x)}{EI} \tag{17}$$

under the assumption that the flexural stiffness EI is constant along the beam.

If the load shows N_C discontinuities, the beam is usually divided in $N_C + 1$, in such a way that the load is continuous in each part; hence, writing and solving equation (17) for each part is necessary. As four boundary conditions have to be considered for each part, a system of $4(N_C + 1)$ algebraic equations has to be solved in order to find the $N_C + 1$ displacement functions. It is evident that, if the beam load presents many discontinuities, this approach becomes onerous from a practical point of view.

The use of the generalised functions introduced in the previous section can overcome the above quoted drawback. For example if the beam load is constant and non-zero only between the abscissas $x = a$ and $x = b$, then it can be represented as follows:

$$p(x) = p[R_0(x-a) - R_0(x-b)] \tag{18}$$

Introducing this expression in equation (17) and performing the integrals (when necessary in generalised sense) up to obtaining the displacement law $u(x)$, the results are:

$$u_{,i}(x) = \frac{p}{EI}[R_{4-i}(x-a) - R_{4-i}(x-b)] + \Delta_i(x), \quad i = 0, 1, 2, 3 \tag{19}$$

where $\Delta_i(x)$, which contains the integration functions and is independent of loading, can be expressed as:

$$\Delta_i(x) = \sum_{j=1}^{4-i} D_j R_{4-j-1}(x) \tag{20}$$

The four constants D_i have to be evaluated by imposing the natural and essential boundary conditions at the beam extremes. It is worth noting that if the usual approach is used the constants to be determined for this example are twelve; while by using the generalised functions they are still four.

Another important example in which the generalised functions can be advantageously used is when the transversal load is a point force F applied at the abscissa x_0. In fact, in this case, the transversal load $p(x)$ can be adequately expressed as follows:

$$p(x) = FR_{-1}(x-x_0) \tag{21}$$

Introducing this expression in equation (17) and performing the integrals in generalised sense up to obtaining the displacement law $u(x)$, we write:

$$u_{,i}(x) = \frac{F}{EI} R_{3-i}(x - x_0) + \Delta_i(x), \quad i = 0, 1, 2, 3 \tag{22}$$

Another important case is when the load is a point moment M applied at the abscissa x_0. In this case, the load $p(x)$ can be adequately expressed by means of a doublet, that is:

$$p(x) = MR_{-2}(x - x_0) \tag{23}$$

Consequently, the generalised integrals give:

$$u_{,i}(x) = \frac{M}{EI} R_{2-i}(x - x_0) + \Delta_i(x), \quad i = 0, 1, 2, 3 \tag{24}$$

It is easy to verify that this case allows us to solve the problems in which the curvature shows some singularities, too. For example, if a beam is subjected to a thermal load, linearly varying along its height and constantly acting between the abscissas $x = a$ and $x = b$, then a jump in the curvature and consequently in the second derivative of the displacement is present. This implies that the fourth-order differential equation governing the displacement law is:

$$u_{,4}(x) = \kappa[R_{-2}(x - a) - R_{-2}(x - b)] \tag{25}$$

κ being the curvature due to the thermal variation. The solution of this equation, can be derived as follows:

$$u_{,i}(x) = \kappa[R_{2-i}(x - a) - R_{2-i}(x - b)] + \Delta_i(x), \quad i = 0, 1, 2, 3 \tag{26}$$

which is a generalization of the expression given in equation (24). In fact, this case is equivalent to the application of two points moments (κEI and $-\kappa EI$) applied at the abscissas $x = a$ and $x = b$, respectively.

In the general case of a combination of N_p uniformly distributed loads p_i acting between the abscissas a_i and b_i, of N_F point forces F_j applied at the abscissas x_j and of N_M point moments M_k at the abscissas x_k, the corresponding displacement law $u(x)$ takes on the following form:

$$\begin{aligned} u(x) = &\sum_{i=1}^{N_p} \frac{p_i}{EI}[R_4(x - a_i) - R_4(x - b_i)] \\ &+ \sum_{j=1}^{N_F} \frac{F_j}{EI} R_3(x - x_j) \\ &+ \sum_{k=1}^{N_M} \frac{M_k}{EI} R_2(x - x_k) + \Delta_4(x) \end{aligned} \tag{27}$$

where the four constants D_i have to be evaluated by imposing the natural and essential conditions at the beam extremes.

DISCONTINUITIES IN THE ROTATION AND DISPLACEMENT

Apart from the cases discussed in the previous section, another case in which the generalised functions can be advantageously used is when some discontinuities on the rotation and/or on the displacement are present in the beam. For example, if a discontinuity in the curvature is located at the abscissa x_0, that is an imposed jump $\Delta\varphi$ in the rotation law is at $x = x_0$. In this case we can write:

$$u_{,1}(x) = \Delta\varphi R_0(x - x_0) \tag{28}$$

that implies:

$$u_{,i}(x) = \Delta p R_{-i}(x - x_0) + \Delta_i(x), \quad i = 0, 1, 2, 3, 4 \tag{29}$$

On the other hand, if an external load induces a located jump Δu in the displacement law at the abscissa x_0, then we can write:

$$u_{,i}(x) = \Delta u R_{-i}(x - x_0) + \Delta_i(x), \quad i = 0, 1, 2, 3, 4 \tag{30}$$

where again the constant D_i has to be evaluated by imposing the boundary conditions.

It is important to note that, for any kind of external loads and for any of their combination, the use of the generalised functions allows us to write a single fourth-order differential equation governing the displacement law. Hence, in any case, the number of constants to be determined is always four.

BEAMS WITH ALONG AXIS ESSENTIAL CONSTRAINTS

In this section and in the following one the results of Brungraber's paper [7] are rearranged and generalised to any kind of essential and natural constraints. In fact the generalised functions can be usefully applied when the beam presents one or more external constraints along its axis, too. For example if a beam is roller supported at the abscissa x_0 and it is loaded by a generic load $p(x)$, then the problem is governed by a differential equation properly expressed as follows:

$$\begin{aligned} u_{,4}(x) &= \frac{p(x)}{EI} + \frac{\hat{F}}{EI} R_{-1}(x - x_0) \\ &\equiv \frac{p(x)}{EI} + \rho_3 R_{-1}(x - x_0) \end{aligned} \tag{31}$$

whose solution gives:

$$u_{,i}(x) = \frac{1}{EI}\underbrace{\int \cdots \int p(x)\,dx}_{4-i \text{ times}} + \rho_3 R_{3-i}(x - x_0) + \Delta_i(x), \quad i = 0,1,2,3 \tag{32}$$

In these equations the value $\hat{F} \equiv \rho_3 EI$ is the roller support reaction; it is unknown, as well as the four constants D_i. The further condition to be imposed is the essential one corresponding to the constrain at x_0, that is $u(x_0) = 0$.

If at x_0 a constrain on the rotation, that is a double-bearing support, is present then the differential equation assumes the following form:

$$u_{,4}(x) = \frac{p(x)}{EI} + \frac{\hat{M}}{EI} R_{-2}(x - x_0) \equiv \frac{p(x)}{EI} + \rho_2 R_{-2}(x - x_0) \tag{33}$$

whose solution is:

$$u_{,i}(x) = \frac{1}{EI}\underbrace{\int \cdots \int p(x)\,dx}_{4-i \text{ times}} + \rho_2 R_{2-i}(x - x_0) + \Delta_i(x), \quad i = 0,1,2,3 \tag{34}$$

In these equations $\hat{M} \equiv \rho_2 EI$ is the unknown reaction of the support. The necessary further condition, behind the natural and essential ones at the extremes, is the essential one $u_{,1}(x_0) = 0$.

Even for this kind of beam it is possible to consider any combination of N_C along axis constraints, and the use of the generalised functions reduce the number of constants to be determined to $N_C + 4$. On the other hand the use of the classical approach needs the evaluation of $4(N_C + 1)$ constants.

BEAMS WITH ALONG AXIS NATURAL CONSTRAINTS

The presence of hinges or bearing joints along the axis of a beam implies a discontinuity in the displacement or rotation law in correspondence of the abscissa x_0 where these internal constraints are. So a further natural condition at the abscissa x_0 arises. For example, if this internal constraint is a hinge, the discontinuity (represented as a unit step function) is related to the rotation, that is to the function $u_{,1}(x)$. Hence, the differential equation governing the problem can be properly written as follows:

$$u_{,4}(x) = \frac{p(x)}{EI} + \Delta\hat{\varphi} R_{-3}(x - x_0) \equiv \frac{p(x)}{EI} + \rho_1 R_{-3}(x - x_0) \tag{35}$$

where $\Delta\hat{\varphi} \equiv \rho_1$ defines the entity of the relative rotation at x_0 and it is a further quantity to be determined, besides the four constant D_i values. The necessary further condition, besides the natural and essential ones at the extremes, is the natural one corresponding to the hinge, that is:

$$u_{,2}(x_0) = 0 \tag{36}$$

The solution of equation (35) is:

$$u_{,i}(x) = \frac{1}{EI}\underbrace{\int \cdots \int p(x)\,dx}_{4-i \text{ times}} + \rho_1 R_{1-i}(x - x_0) + \Delta_i(x), \quad i = 0,1,2,3 \tag{37}$$

If the internal constraint at x_0 is a bearing joint, then the discontinuity is on the displacement and the differential equation takes the form:

$$u_{,i}(x) = \frac{p(x)}{EI} + \Delta\hat{u} R_{-4}(x - x_0) \equiv \frac{p(x)}{EI} + \rho_0 R_{-4}(x - x_0) \tag{38}$$

where $\Delta\hat{u} \equiv \rho_0$ is the unknown entity of the relative displacement at x_0. The solution of equation (38) is:

$$u_{,i}(x) = \frac{1}{EI}\underbrace{\int \cdots \int p(x)\,dx}_{4-i \text{ times}} + \rho_0 R_{0-i}(x - x_0) + \Delta_i(x) \quad i = 0,1,2,3 \tag{39}$$

and the further natural condition is:

$$u_{,3}(x_0) = 0 \tag{40}$$

It is obvious that any combination of these internal constraints can be present, placed at the same or different abscissas, making even more advantageous the use of the improper functions respect to the classical approach.

BEAMS WITH ALONG AXIS MIXED CONSTRAINTS

In the sections 5 and 6 the cases of beams with along axis essential and natural constraints, respectively, have been considered. But, in some cases, the constraint can be considered as a mixed type one. For example, this happens when the beam presents a spring roller support at the abscissa x_0. In this case, the differential equation governing the problem can be properly written as follows:

$$u_{,4}(x) = \frac{p(x)}{EI} + \rho_3 R_{-1}(x - x_0) \tag{41}$$

whose solution is given by:

$$u_{,i}(x) = \frac{1}{EI}\underbrace{\int\cdots\int}_{4-i\text{ times}} p(x)\,dx + \rho_3 R_{3-i}(x - x_0) + \Delta_i(x), \quad i = 0, 1, 2, 3 \tag{42}$$

Except the natural and essential conditions, the necessary further condition for the evaluation of the four constants D_i and the spring roller reaction $\hat{F} \equiv \rho_3 EI$ is of mixed type, that is:

$$\rho_3 = -\frac{K}{EI}u(x_0) \tag{43}$$

K being the stiffness of the spring.

The case of a beam with a spring double-bearing support is very similar to the previous one. In fact, the equation governing the problem is equal to equation (33), whose solution is given in equation (34). The further condition to be imposed is the following mixed one:

$$\rho_2 = -\frac{K_\varphi}{EI}u_{,1}(x_0) \tag{44}$$

K_φ being the stiffness of the angular spring.

Mixed type constraints arise when spring hinges and spring bearing joints are present along the beam axis, too. In fact, if a spring hinge is located at the abscissa x_0, the beam bending is governed by equation (35). The corresponding solution is given in equation (37). The further mixed condition, necessary for the evaluation of the relative rotation $\Delta\hat{\varphi} \equiv \rho_1$, is:

$$\rho_1 = \frac{EI}{K_\varphi}u_{,2}(x_0) \tag{45}$$

At last, when a spring bearing joint is located at the abscissa x_0, the differential equation governing the problem has the form of equation (38), whose solution is given in equation (39). The further mixed condition, necessary for the evaluation of the relative displacement $\Delta\hat{u} \equiv \rho_0$, is:

$$\rho_0 = \frac{EI}{K}u_{,3}(x_0) \tag{46}$$

At the end of these last three sections, it is important to note that, for any kind of discontinuities considered, it is always possible to write the differential equation governing the beam bending problem in the following form:

$$u_{,i}(x) = \frac{p(x)}{EI} + \rho_j R_{j-4}(x - x_0), \quad j = 0, 1, 2, 3 \tag{47}$$

The case $j = 0$ refers to the discontinuity on the displacements, which can be known, as in equation (30), or unknown, as in equation (38). The case $j = 1$ refers to the case of discontinuity in the rotations; it can be known, as in equation (29), or unknown, as in equation (35). The case $j = 2$ refers to the discontinuity on the moments; it can be known, as in equation (24), or unknown, as in equation (33). At last, the case $j = 3$ refers to the discontinuity of shear forces; it can be known, as in the case of equation (22), or unknown, as in the case of equation (31). In any case, the solution of equation (47) can be written as follows:

$$u_{,i}(x) = \frac{1}{EI}\underbrace{\int\cdots\int}_{4-i\text{ times}} p(x)\,dx + \rho_j R_{j-1}(x - x_0) + \Delta_i(x), \quad i = 0, 1, 2, 3, \quad j = 0, 1, 2, 3 \tag{48}$$

If the discontinuity is known, only the four natural and essential conditions at the beam extreme have to be considered in order to evaluate the four constant D_i values. If the discontinuity is unknown, a further condition has to be considered in order to evaluate it. This condition can be of natural type (for $j = 0$, 1), or of essential type (for $j = 2$, 3), or of mixed type, in the case of spring constraints.

EXAMPLE

As an example, the beam represented in Fig. 1 is taken into account. If the classical approach is used, five laws of displacements have to be obtained; consequently twenty constants have to be evaluated by imposing twenty boundary conditions on the extremes of each of the five parts in which the beam is divided. On the other hand, if the generalised functions are used, the differential equation governing the problem is expressed as follows:

$$u_{,4}(x) = \frac{1}{EI}[pR_0(x - 4l) + FR_{-1}(x - l)] + \rho_2 R_{-2}(x - 3l) + \rho_1 R_{-3}(x - 2l) \tag{49}$$

whose formal integrals give:

$$u_{,i}(x) = \frac{1}{EI}[pR_{4-i}(x - 4l) + FR_{3-i}(x - l)] + \rho_2 R_{2-i}(x - 3l) + \rho_1 R_{1-i}(x - 2l) + \Delta_i(x) \tag{50}$$

The six quantities D_i, ρ_1 and ρ_2 have to be evaluated by imposing the boundary conditions at the extremes and in correspondence of the hinge

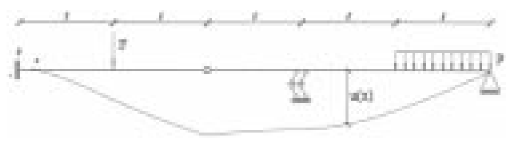

Fig. 1. Beam of the example and corresponding displacement diagram.

and of the external double-bearing support; they are essential type one, that is:

$$u(0) = 0; \quad u'(0) = 0; \quad u(5l) = 0; \quad u'(3l) = 0 \tag{51}$$

and natural type one:

$$u''(5l) = 0; \quad u''(2l) = 0 \tag{52}$$

The value of these six constants are:

$$\begin{aligned} D_1 &= -\frac{11}{34}\frac{F}{EI} - \frac{23}{136}\frac{pl}{EI}; \\ D_2 &= -\frac{6}{17}\frac{Fl}{EI} + \frac{23}{68}\frac{pl^2}{EI}; \\ D_3 &= D_4 = 0; \\ \rho_2 &= -\frac{69}{34}\frac{Fl}{EI} + \frac{1}{136}\frac{pl^2}{EI}; \\ \rho_1 &= \frac{103}{68}\frac{Fl^2}{EI} - \frac{69}{272}\frac{pl^4}{EI} \end{aligned} \tag{53}$$

They have been evaluated without any computing support. In Fig.1 the elastic displacement law $u(x)$ is represented in the case in which $F = pl$, too. This graphic can be easily obtained by means of any symbolic mathematical code.

CONCLUSIONS

In this paper the problem of the beam bending differential equations showing any kind of discontinuity is presented. The method used can be considered as an extension of the Macaulay's approach, which takes into account the discontinuities in the external mechanical loads, that is the presence of point forces and moments. This method uses the properties of the generalised functions, as the unit step function or the Dirac delta function and all the functions that can be derived as the formal derivative and integral of these. In the paper it is explained in what mathematical sense these formal operations have to be considered. At this purpose, it is important to note that the generalised functions can be usefully considered in any field of the *studiorum curriculum* of an engineer student in which some discontinuities in the governing equations of a problem have to take into account. For this the author thinks that the generalised functions represent an *undeletable* subject in the education of an engineer. In the framework of the beam bending problem, the use of the generalised functions has allowed us to obtain a uniform treatment for any kind of discontinuity that can arise. Moreover, it has been shown that the use of this method reduces the computational effort in the evaluation of the integration constants. In fact, their number is always four, if the value of the discontinuities is known, or $N_C + 4$, N_C being the number of unknown discontinuities characterising the problem. The algebraic equations determining these constants are the four boundary natural and essential conditions and the N_C conditions related to the along-axis constraints. These last ones can be of natural, essential and/or mixed type, depending on the kind of discontinuity. The number $N_C + 4$ has to be compared with $4(N_C + 1)$, which is the number of integration constants to be determined if the classical approach of the beam bending differential equations is considered.

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