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Visual Instruction of Abstract Concepts in Mathematics Courses for Non-major Students*

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Abstract mathematics courses have been difficult for many non-major students including engineering students. This article discusses the results of a study intended to improve students' understanding of abstract concepts in mathematics courses, and to better prepare students for advanced courses in disciplines such as engineering and, as a result, increase retention rate. The study implemented Mathematica, a computer algebra system (CAS), as a visual aid in learning basic linear algebra concepts. Overall, the results supported the role of visual demonstrations/representations in advancing students' understanding of abstract concepts.

INTRODUCTION

VALKENBURG [1] stated, 'Improving the mathematics preparation of our students will significantly improve the overall effectiveness of our undergraduate programs', in an article about the role of mathematics in preparing engineering students, and in improving effectiveness of engineering programs. The majority of us, shareholders of the responsibility of the education of engineering students, would agree with Valkenburg that mathematics plays a crucial role in engineering students' success in engineering programs, and that mathematics is one of the primary vehicles used in preparing critical thinkers and independent learners of the future in engineering.

Due to advances in technologies, such as digital computers, that are used widely in engineering schools [2–4], and to the use of linear algebra concepts in these technologies, linear algebra and matrix algebra are among the advanced mathematics courses attracting more and more students from other disciplines, especially, from engineering [5].

Non-major students including engineering students are usually not prepared or at best ill-prepared for the high abstraction level of linear and matrix algebra courses: they are so lost in much of the abstraction of concepts that even the simplest ideas become difficult to comprehend, and this often leads to discouragement, high stress, 'burn out,' and, as a result, high failure rate. In 2001, for instance, at a four-year southwestern US research university, 25% withdrew and 34% failed matrix algebra, a first abstract course offered for non-majors. Typical comments, taken from the faculty evaluations of the students enrolled in these courses, expressed feeling lost in symbols, notations and abstraction, and, as a result, feeling discouraged from trying to make further sense of topics covered.

Learning difficulties occurring due to the growing heterogeneity of US linear (matrix) algebra classes, brought up the question of how one can modify a 'first year linear algebra curriculum' so that it can respond to the needs of both mathematics and non-mathematics students including engineering students. This led to the Linear Algebra Curriculum Study Group recommendations for the first course in linear algebra [6–8]. The recommendations made by the group, in summary, were:

1. The syllabus and presentation of the first course in linear algebra must respond to the needs of client disciplines such as engineering.
2. Mathematics departments should seriously consider making their first course in linear algebra a matrix-oriented course.

A few studies have investigated the learning difficulties occurring in linear (matrix) algebra classrooms, most of which [9–12] reported difficulties with abstraction level of the course material; in recognizing different representations of the same concepts; and the lack of logic and set theory knowledge [13]. According to Dubinsky [14] and Harel [15], students can achieve abstraction if flexibility between representations of the same concepts is established. They also believed that abstraction might be established if concept images, defined as all mental pictures, properties and processes associated with the concept, and concept definitions, defined as a form of symbols used to specify the concept, are not contradicting one another. Then the use of *visual* instruction

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might help students have better mental images and, as a result, deeper understanding. This study implemented visual instruction via the use of *Mathematica*^{TM} notebooks (a computer algebra system (CAS)), and investigated its effect on the understanding of abstract concepts.

Compared to the wide range of computer activities (mostly visual-based) [16, 17] such as those of the ATLAST project [18] used in teaching first-year linear (matrix) algebra concepts, little has been done to investigate the effect of use of technology and visual instruction on learning and teaching. Sierpinska et al. [19] discussed the effect of Cabri (a dynamic geometry software [20]) on students' mental images of both linear combination and linear independence. Their instruction was based on a geometric model of two-dimensional vector spaces within the dynamic Cabri-geometry II environment. Their study indicated no significant findings. On the other hand, Leron and Dubinsky [21] reported, as a result of writing programs in ISETL (a programming language [22]) as the solutions for abstract algebra questions, substantial increase in student's understanding of abstract algebra concepts. The computer programs were not visual-oriented, but allowed students, through experimentation, to investigate, discover, and construct their own understanding.

PURPOSE OF THE STUDY

There is a need for an in-depth study to better understand how visual demonstrations/representations can help students, especially non-major students with no or little preparation of abstract thinking, cope with the abstraction level of advanced mathematical concepts. Little has been done to investigate the effect. This study, through the use of *Mathematica* notebook visual demonstrations, is one that proposed to test the effect through a comparative design.

METHOD

Data collection

Data were collected from two fall 1999 first-year linear algebra classes taught at a four-year US research university. One of the courses was taught traditionally by a professor of mathematics, and the other was taught by the investigator in a laboratory environment with the use of *Mathematica*^{TM} notebooks covering two- and three-dimensional visual demonstrations of basic abstract linear algebra concepts. Both sections met two times a week. The traditional group met from 1 p.m. to 2:20 p.m., and the experimental group met from 9 a.m. to 10:20 p.m. Due to the limitations to course scheduling, it was not possible to have sections offered during the same hours. Efforts, however, were made to ensure that the two classes were equal in every aspect of instruction except the experimental aspect. That is, both classes covered the same material including examples during the class meetings of the same week, and assigned the same set of homework questions. The difference was the way materials such as examples were introduced and discussed (also, occasionally, the way homework questions were discussed). In the experimental group, all materials were introduced and discussed, first, via visual demonstrations, and next, via formal instruction. In the traditional group, on the other hand, formal instruction was the only means used to introduce and discuss the course material.

Data collection included a background questionnaire, a pre-test, in-class observations, and five post-test questions (see Appendix for the questions) as well as a post-questionnaire. In addition, in an attempt to get better insight on students' responses on the five post-questions, interviews were conducted with twelve volunteers towards the end of the semester.

Participants

Participants were students enrolled in two sections of first-year linear algebra classes in the fall of 1999. Students voluntarily participated in the study in the sense that even though they had the option of switching to one of the other two linear algebra classes that were not used in the study, none switched sections. Both classes were regular-size classes with 29 students in the experimental group, and 35 in the traditional group. Ninety percent of the traditional group, and seventy percent of the experimental group consisted of engineering students. The experimental group also had many students from economics (30%). Before enrolling in linear algebra, approximately 13% of the traditional group and 9% of the experimental group have taken one to two advanced math courses (courses after calculus) that may have fully or partially prepared them for abstract thinking.

Data analysis

Quantitative and qualitative research methods were chosen in an attempt both to determine any significance of the results, and to discover students' thinking patterns, strengths and weakness. Since the sample sizes were not equal, an Aspin-Welch-Satterthwaite (AWS) t' statistic [23] with the assumption of unequal variances was applied to test the null hypothesis that there is no difference between the students' mean scores on post-test questions. Furthermore, to test whether or not the treatment explained a significant amount of the variability, a non-orthogonal two-way analysis of variance adjusting for the control variables (attendance, nationality, gender, and ability) and ignoring the interaction variables was applied—the independent variables were treatment and control variables [24]. Grading of each post-question was done by the investigator and by four graduate

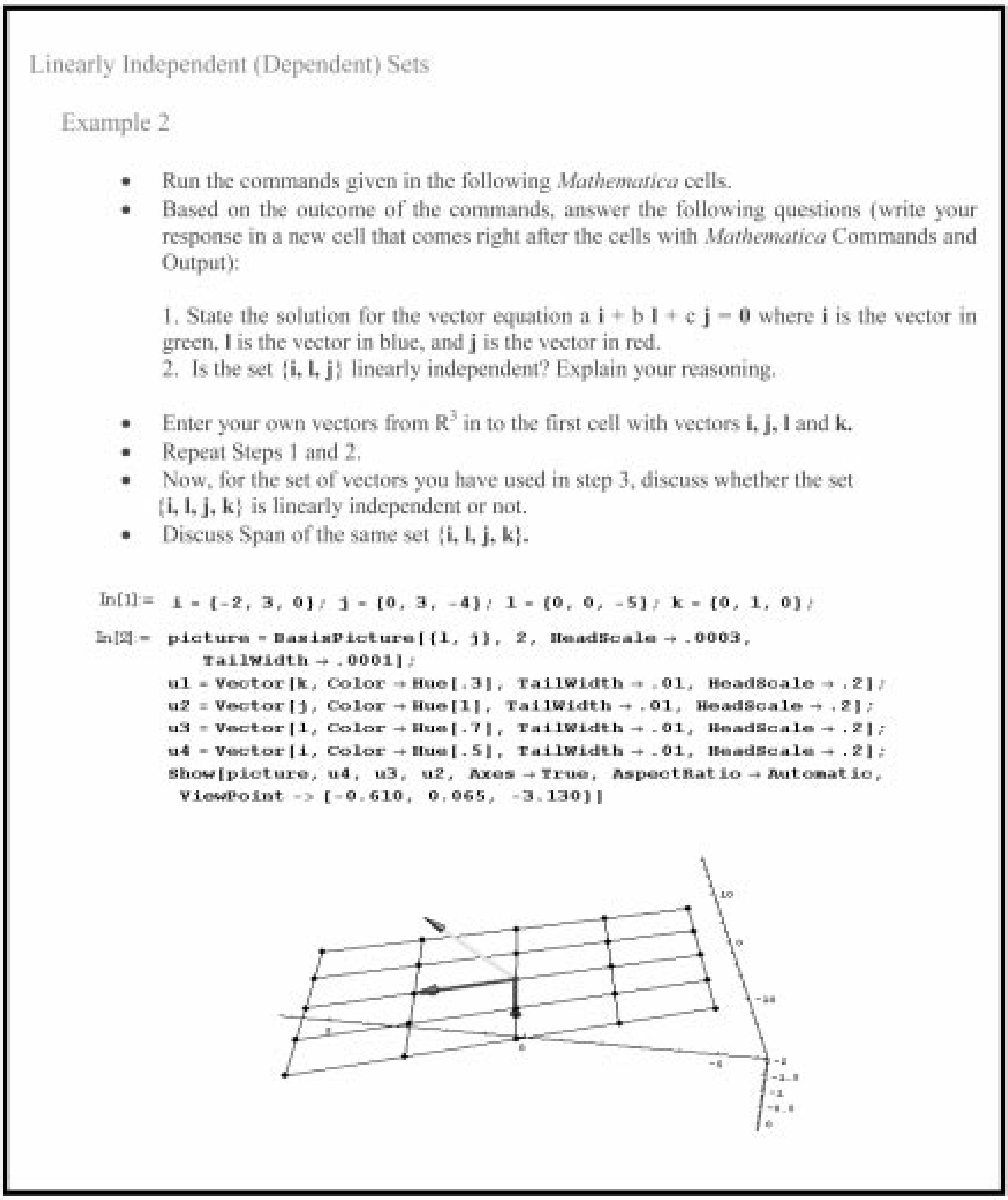

Fig. 1. *Mathematica*^{TM} notebook addressing linearly independent (dependent) vectors, span and spanning set. This shows how the notebook looks after running *Mathematica*^{TM} commands. Titles are written in red, and instruction is written in blue. Some of the *Mathematica*^{TM} commands used in this notebook are modified from Wicks [17].

students from the mathematics department. To maintain consistency among graders, a 5-point rubric [25] was used. Reliability between the graders was found to be ranging from 0.79 to 0.90.

In this paper, since the results of the non-orthogonal two-way analysis of variance are not significant, only the results of AWS t′ statistic along with the results of the post-questionnaire and the interviews are reported.

Mathematica notebooks

Mathematica^{TM} notebooks containing *Mathematica*^{TM} commands, some of which were modified from the textbook by Wicks [17], were written by the investigator as interactive, guided supplements to lectures. They were primarily composed of interactive cells of examples of basic linear algebra concepts. Figure 1 provides an example of such a notebook. Emphases were given to two- and three-dimensional *visual* demonstrations of

Table 1. Results of students' scores on the post-questions (see Appendix for the questions). Results are summarized under three categories. Here 'S' indicates statistical significance, and 'NS' indicates no significance. 'E' stands for experimental group, and 'T' stands for traditional group.

	Sample size		Mean		SD		
Categories—content (Question number)	E	T	E	T	E	T	AWS t′ test
II—Subspace (Q1a)	24	31	3.35	3.77	1.37	1.09	NS
I—Basis (Q1b)	23	31	3.09	2.09	1.56	1.31	S
III—Linear independence (Q2i)	25	26	3.97	3.80	1.36	1.44	NS
III—Dimension, linear independence and span (Q2ii)	25	26	2.70	2.90	1.55	1.44	NS
III—Span (Q2iii)	25	26	3.24	3.50	1.58	1.27	NS
II—Linear independence (Q3a)	26	28	3.84	3.75	1.31	1.35	NS
II—Spanning set (Q3b)	25	28	3.10	3.42	1.39	1.50	NS
III—Basis and linear transformation (Q4a)	22	28	3.00	3.20	1.34	1.19	NS
I—Dimension and linear transformation (Q4b)	22	28	2.59	2.07	0.73	1.08	S
I—Span and linear independence (proof) (Q5)	26	28	3.12	3.00	1.53	1.71	NS

basic vector space concepts. Each cell in a notebook contained an example discussed in class, and was labeled accordingly.

Before the introduction of formal (abstract) definitions, related examples from *Mathematica*^{TM} cells were run in class, and class discussions of the outcomes took place. As more similar interactive cells with different examples of the same concepts were run and discussed, students were asked to write their own interpretations into the *Mathematica* cell that comes right after the cells with the *Mathematica*^{TM} commands and the *Mathematica*^{TM} output. Students were to answer questions (see Fig. 1) through analyzing visual *Mathematica*^{TM} outputs.

For instance, *Mathematica*^{TM} notebooks similar to the one in Fig. 1 were used to discuss linear independence, and its connection to the concepts of span, spanning sets, and bases. These activities were mainly used to help students gain deeper understanding of the formal (abstract) definition of a linearly independent set stated on the textbook by Larson and Edwards [26] as:

> A set of vectors $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\}$ in a vector space V is called **linearly independent** if the vector equation $c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \ldots + c_k\boldsymbol{v}_k = \boldsymbol{0}$ has only the trivial solution, $c_1 = 0, c_2 = 0, \ldots, c_k = 0$. If there are also nontrivial solutions, then S is called **linearly dependent.**

RESULTS AND DISCUSSION

Table 1 summarizes the results of the AWS t′ analysis of students' responses on the five post-questions (see appendix for the questions). Questions in Table 1 labeled as Category I are categorized as those requiring conceptual knowledge, defined as knowledge that is rich in relationships [27], and those labeled as Category II requiring procedural knowledge, defined as knowledge of symbols and syntax of mathematics that implies only an awareness of surface features, not knowledge of meaning [27]. Once the investigator completed interviews, another category emerged. This new category called Category III consists of questions similar to examples that had been discussed during class time in both sections. Findings from the interviews hinted a relation between students' recall of examples covered in class and their performance on the Category III type questions. During the interviews, students from both groups showed tendency to mention examples introduced in class, and to apply the procedures used in these examples to the interview questions. In light of the interview results, one might assume that the differences shown in Table 1 for Category III type questions may not be due to the treatment, but due to students' ability to recall examples discussed in class.

Table 1 shows a statistically significant difference between the two groups, favoring the experimental group, on two of the three Category I type questions. No significant difference is observed on the questions of Category II or III types. On the last question (Category I type) where students were asked to write a proof, even though there has been no significance observed, one must agree that the experimental group performed slightly better. Research [28, 29] has reported that proof writing may require more than what the technology used in this study offered: knowing how to begin a proof is one of the requirements reported by Moore [29]. One might then conclude that visual-based computer aid alone may not be enough to gain significant improvement in students' proof writing skills.

The implementation of *Mathematica*^{TM} overall seemed to have a positive effect on students' motivation. Compared to the traditional group's opinion (50%), significantly more students in the experimental group (70%) indicated enjoyment with the class. This seemed to have led to higher retention of students in the experimental group—as opposed to 70% of the students enrolled in the traditional group, 90% of those enrolled in the experimental group completed the course. The interviews also suggested possible long-term effects of visual *Mathematica*^{TM} activities in remembering basic abstract concepts: the students in the experimental group claimed that they would remember basic definitions in long-term whereas the traditional group could not remember even the definitions during the interviews, even though they had an exam the next day. They indicated that they would, the night before the exam, sit down and memorize the definitions. The statement shown below represents typical responses obtained from the traditional group to the definition-related interview questions:

> In physics I was given a very restricted definition of a vector. . . [my instructor] has been discussing a very
>
> general abstract definition of a vector so . . . reading the linear algebra book tonight [an exam was scheduled for the next day], I will think of [my instructor]'s definition but up till now I think of the physics definition that I have got.

In short, the interviews and the results of the analysis of the post questions indicated that the students in the experimental group had higher conceptual understanding of basic abstract linear algebra concepts and definitions than those of the traditional group, and performed equally well on the procedural questions.

CONCLUSION

This paper attempted to discuss the effect of use of *Mathematica* as a visual aid in introducing basic linear algebra concepts. Overall, considering the nature of the student population (the majority were non-major students) of the two courses (engineering students (90%) in the traditional group, and (70%) in the experimental group) used in the study, the findings support the use of visual instruction in advancing the learning of abstract concepts, especially for students with limited prior preparation for abstract thinking.

One of the implications of the study reported in this paper is that visual instruction may reduce abstraction-related learning obstacles in mathematics courses that are serving mostly non-majors such as engineers. This may result in increased motivation and a higher success rate leading to a higher retention rate. This may also mean mathematically better-prepared students for more advanced engineering courses, with significant improvement of the overall effectiveness of undergraduate engineering programs [1].

In light of the findings and implications reported in this paper, I would recommend for instructors of mathematics courses serving mainly non-majors such as engineers to consider implementing *visual-based* instruction to help their students overcome obstacles to learning abstract concepts.

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APPENDIX

Post questions

Q1: Let S be the set of matrices of the form:

$$\begin{bmatrix} a & b \\ b & a \end{bmatrix}$$

where a and b are any real numbers.

(a) Show that S is a subspace of $M_{2,2}$
(b) Find a basis for S.

Q2: Given the following vectors: R^3.

a = (0, –1, 1), b = (2, 1, 1), c = (2, 0, 2), d = (1, 0, 1)

i. Is the set {a, b, c} linearly independent (justify your answer)?
ii. What is the dimension of Span {a, b, c} (justify your answer)?
iii. Is the vector (1, 2, 3) in Span {a, b, c} (justify your answer)?

Q3: Define the following terms, and give an example for each term:

a. Linearly independent set
b. Spanning set

Q4: Given a linear transformation T: $R^3 \to R^2$ by T(v) = Av where:

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

a. Find a basis for Ker (T).
b. What is the dim (Image (T)) (Justify your answer)?

Q5: Suppose that Span $\{v_1, v_2, v_3, \ldots, v_n\} = V$ and w is a vector in V:

Prove or disprove that the set $\{v_1, v_2, v_3, .., v_n, w\}$ is linearly dependent.

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